THE HARVARD-MIT MATHEMATICS TOURNAMENT

ALGEBRA TEST

This test consists of 10 short-answer problems to be solved individually in 50 minutes. Problems will be weighted with point values after the contest based on how many competitors solve each problem. There is no penalty for guessing.

No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.

Answers should be simplified as much as is reasonably possible and must be exact unless otherwise specified. Rational numbers should be written in lowest terms, although denominators of irrationals need not be rationalized. An nth root should be simplified so that the radicand is not divisible by the nth power of any prime.

Correct mathematical notation must be used. No partial credit will be given unless otherwise specified.

If you believe the test contains an error, please submit your protest in writing to Lobby 10 during lunchtime.

Enjoy!

HMMT 2013

Saturday 16 February 2013

Algebra Test

1. Let x and y be real numbers with $x > y$ such that $x^2y^2 + x^2 + y^2 + 2xy = 40$ and $xy + x + y = 8$. Find the value of x.

2. Let $\{a_n\}_{n\geq 1}$ be an arithmetic sequence and $\{g_n\}_{n\geq 1}$ be a geometric sequence such that the first four terms of $\{a_n + g_n\}$ are 0, 0, 1, and 0, in that order. What is the 10th term of $\{a_n + g_n\}$?

3. Let S be the set of integers of the form $2^x + 2^y + 2^z$, where x, y, z are pairwise distinct non-negative integers. Determine the 100th smallest element of S.

4. Determine all real values of A for which there exist distinct complex numbers x_1, x_2 such that the following three equations hold:

$$x_1(x_1 + 1) = A$$
$$x_2(x_2 + 1) = A$$
$$x_1^4 + 3x_1^3 + 5x_1 = x_2^4 + 3x_2^3 + 5x_2.$$

5. Let a and b be real numbers, and let r, s, and t be the roots of $f(x) = x^3 + ax^2 + bx - 1$. Also, $g(x) = x^3 + mx^2 + nx + p$ has roots r^2, s^2, and t^2. If $g(-1) = -5$, find the maximum possible value of b.

6. Find the number of integers n such that

$$1 + \left\lfloor \frac{100n}{101} \right\rfloor = \left\lceil \frac{99n}{100} \right\rceil.$$

7. Compute

$$\sum_{a_1=0}^{\infty} \sum_{a_2=0}^{\infty} \cdots \sum_{a_7=0}^{\infty} \frac{a_1 + a_2 + \cdots + a_7}{3^{a_1+a_2+\cdots+a_7}}.$$

8. Let x, y be complex numbers such that $\frac{x^2+y^2}{x+y} = 4$ and $\frac{x^4+y^4}{x^3+y^3} = 2$. Find all possible values of $\frac{x^6+y^6}{x^5+y^5}$.

9. Let z be a non-real complex number with $z^{23} = 1$. Compute

$$\sum_{k=0}^{22} \frac{1}{1 + z^k + z^{2k}}.$$

10. Let N be a positive integer whose decimal representation contains 11235 as a contiguous substring, and let k be a positive integer such that $10^k > N$. Find the minimum possible value of

$$\frac{10^k - 1}{\gcd(N, 10^k - 1)}.$$

HMMT 2013
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Algebra Test

PUT LABEL HERE

Name ________________________________ Team ID# ________

Organization ________________________ Team ____________________

1. ________________
2. ________________
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10. ________________

Score: ____________