HMMT 2013

Saturday 16 February 2013

Algebra Test

1. Let x and y be real numbers with $x > y$ such that $x^2y^2 + x^2 + y^2 + 2xy = 40$ and $xy + x + y = 8$. Find the value of x.

   Answer: $\boxed{3+\sqrt{7}}$ We have $(xy)^2 + (x+y)^2 = 40$ and $xy + (x+y) = 8$. Squaring the second equation and subtracting the first gives $xy(x+y) = 12$ so $xy, x+y$ are the roots of the quadratic $a^2 - 8a + 12 = 0$. It follows that $\{xy, x+y\} = \{2, 6\}$. If $x+y = 2$ and $xy = 6$, then x, y are the roots of the quadratic $b^2 - 2b + 6 = 0$, which are non-real, so in fact $x+y = 6$ and $xy = 2$, and x, y are the roots of the quadratic $b^2 - 6b + 2 = 0$. Because $x > y$, we take the larger root, which is $\frac{6+\sqrt{28}}{2} = 3 + \sqrt{7}$.

2. Let $\{a_n\}_{n\geq 1}$ be an arithmetic sequence and $\{g_n\}_{n\geq 1}$ be a geometric sequence such that the first four terms of $\{a_n + g_n\}$ are 0, 0, 1, and 0, in that order. What is the 10th term of $\{a_n + g_n\}$?

   Answer: $\boxed{-54}$ Let the terms of the geometric sequence be a, ra, r^2a, r^3a. Then, the terms of the arithmetic sequence are $-a, -ra, -r^2a + 1, -r^3a$. However, if the first two terms of this sequence are $-a, -ra$, the next two terms must also be $(-2r+1)a, (-3r+2)a$. It is clear that $a \neq 0$ because $a_3 + g_3 \neq 0$, so $-r^3 = -3r + 2 \Rightarrow r = 1$ or -2. However, we see from the arithmetic sequence that $r = 1$ is impossible, so $r = -2$. Finally, by considering a_3, we see that $-4a + 1 = 5a$, so $a = 1/9$. We also see that $a_n = (3n-4)a$ and $g_n = (-2)^{n-1}a$, so our answer is $a_{10} + g_{10} = (26 - 512)a = -486a = -54$.

3. Let S be the set of integers of the form $2^x + 2^y + 2^z$, where x, y, z are pairwise distinct non-negative integers. Determine the 100th smallest element of S.

   Answer: $\boxed{577}$ S is the set of positive integers with exactly three ones in its binary representation. The number of such integers with at most d total bits is $\binom{d}{3}$, and noting that $\binom{9}{3} = 84$ and $\binom{10}{3} = 120$, we want the 16th smallest integer of the form $2^9 + 2^x + 2^y$, where $y < x < 9$. Ignoring the 2^9 term, there are $\binom{d'}{2}$ positive integers of the form $2^x + 2^y$ with at most d' total bits. Because $\binom{6}{2} = 15$, our answer is $2^9 + 2^6 + 2^0 = 577$. (By a *bit*, we mean a digit in base 2.)

4. Determine all real values of A for which there exist distinct complex numbers x_1, x_2 such that the following three equations hold:

$$x_1(x_1+1) = A$$
$$x_2(x_2+1) = A$$
$$x_1^4 + 3x_1^3 + 5x_1 = x_2^4 + 3x_2^3 + 5x_2.$$

   Answer: $\boxed{-7}$ Applying polynomial division,

$$\begin{aligned} x_1^4 + 3x_1^3 + 5x_1 &= (x_1^2 + x_1 - A)(x_1^2 + 2x_1 + (A-2)) + (A+7)x_1 + A(A-2) \\ &= (A+7)x_1 + A(A-2). \end{aligned}$$

   Thus, in order for the last equation to hold, we need $(A+7)x_1 = (A+7)x_2$, from which it follows that $A = -7$. These steps are reversible, so $A = -7$ indeed satisfies the needed condition.

5. Let a and b be real numbers, and let r, s, and t be the roots of $f(x) = x^3 + ax^2 + bx - 1$. Also, $g(x) = x^3 + mx^2 + nx + p$ has roots r^2, s^2, and t^2. If $g(-1) = -5$, find the maximum possible value of b.

   Answer: $\boxed{1+\sqrt{5}}$ By Vieta's Formulae, $m = -(r^2 + s^2 + t^2) = -a^2 + 2b$, $n = r^2s^2 + s^2t^2 + t^2r^2 = b^2 + 2a$, and $p = -1$. Therefore, $g(-1) = -1 - a^2 + 2b - b^2 - 2a - 1 = -5 \Leftrightarrow (a+1)^2 + (b-1)^2 = 5$. This is an equation of a circle, so b reaches its maximum when $a + 1 = 0 \Rightarrow a = -1$. When $a = -1$, $b = 1 \pm \sqrt{5}$, so the maximum is $1 + \sqrt{5}$.

6. Find the number of integers n such that

$$1+\left\lfloor \frac{100n}{101} \right\rfloor = \left\lceil \frac{99n}{100} \right\rceil.$$

Answer: $\boxed{10100}$ Consider $f(n) = \lceil \frac{99n}{100} \rceil - \lfloor \frac{100n}{101} \rfloor$. Note that $f(n+10100) = \lceil \frac{99n}{100} + 99\cdot 101 \rceil - \lfloor \frac{100n}{101} + 100^2 \rfloor = f(n) + 99\cdot 101 - 100^2 = f(n) - 1$. Thus, for each residue class r modulo 10100, there is exactly one value of n for which $f(n) = 1$ and $n \equiv r \pmod{10100}$. It follows immediately that the answer is 10100.

7. Compute

$$\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_7=0}^{\infty}\frac{a_1+a_2+\cdots+a_7}{3^{a_1+a_2+\cdots+a_7}}.$$

Answer: $\boxed{15309/256}$ Note that, since this is symmetric in a_1 through a_7,

$$\begin{aligned}\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_7=0}^{\infty}\frac{a_1+a_2+\cdots+a_7}{3^{a_1+a_2+\cdots+a_7}} &= 7\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_7=0}^{\infty}\frac{a_1}{3^{a_1+a_2+\cdots+a_7}} \\ &= 7\left(\sum_{a_1=0}^{\infty}\frac{a_1}{3^{a_1}}\right)\left(\sum_{a=0}^{\infty}\frac{1}{3^a}\right)^6.\end{aligned}$$

If $S = \sum \frac{a}{3^a}$, then $3S - S = \sum \frac{1}{3^a} = 3/2$, so $S = 3/4$. It follows that the answer equals $7\cdot\frac{3}{4}\cdot\left(\frac{3}{2}\right)^6 = \frac{15309}{256}$. Alternatively, let $f(z) = \sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_7=0}^{\infty} z^{a_1+a_2+\cdots+a_7}$. Note that we can rewrite $f(z) = \left(\sum_{a=0}^{\infty} z^a\right)^7 = \frac{1}{(1-z)^7}$. Furthermore, note that $zf'(z) = \sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_7=0}^{\infty}(a_1+a_2+\cdots+a_7)z^{a_1+a_2+\cdots+a_7}$, so the sum in question is simply $\frac{f'(1/3)}{3}$. Since $f'(x) = \frac{7}{(1-z)^8}$, it follows that the sum is equal to $\frac{7\cdot 3^7}{2^8} = \frac{15309}{256}$.

8. Let x, y be complex numbers such that $\frac{x^2+y^2}{x+y} = 4$ and $\frac{x^4+y^4}{x^3+y^3} = 2$. Find all possible values of $\frac{x^6+y^6}{x^5+y^5}$.

Answer: $\boxed{10 \pm 2\sqrt{17}}$ Let $A = \frac{1}{x} + \frac{1}{y}$ and let $B = \frac{x}{y} + \frac{y}{x}$. Then

$$\frac{B}{A} = \frac{x^2+y^2}{x+y} = 4,$$

so $B = 4A$. Next, note that

$$B^2 - 2 = \frac{x^4+y^4}{x^2y^2} \quad\text{and}\quad AB - A = \frac{x^3+y^3}{x^2y^2},$$

so

$$\frac{B^2-2}{AB-A} = 2.$$

Substituting $B = 4A$ and simplifying, we find that $4A^2 + A - 1 = 0$, so $A = \frac{-1\pm\sqrt{17}}{8}$. Finally, note that

$$64A^3 - 12A = B^3 - 3B = \frac{x^6+y^6}{x^3y^3} \quad\text{and}\quad 16A^3 - 4A^2 - A = A(B^2-2) - (AB - A) = \frac{x^5+y^5}{x^3y^3},$$

so

$$\frac{x^6+y^6}{x^5+y^5} = \frac{64A^2-12}{16A^2-4A-1} = \frac{4-16A}{3-8A},$$

where the last inequality follows from the fact that $4A^2 = 1-A$. If $A = \frac{-1+\sqrt{17}}{8}$, then this value equals $10+2\sqrt{17}$. Similarly, if $A = \frac{-1-\sqrt{17}}{8}$, then this value equals $10-2\sqrt{17}$.

(It is not hard to see that these values are achievable by noting that with the values of A and B we can solve for $x+y$ and xy, and thus for x and y.)

9. Let z be a non-real complex number with $z^{23}=1$. Compute

$$\sum_{k=0}^{22} \frac{1}{1+z^k+z^{2k}}.$$

Answer: $\boxed{46/3}$ *First solution:* Note that

$$\sum_{k=0}^{22} \frac{1}{1+z^k+z^{2k}} = \frac{1}{3} + \sum_{k=1}^{22} \frac{1-z^k}{1-z^{3k}} = \frac{1}{3} + \sum_{k=1}^{22} \frac{1-(z^{24})^k}{1-z^{3k}} = \frac{1}{3} + \sum_{k=1}^{22}\sum_{\ell=0}^{7} z^{3k\ell}.$$

3 and 23 are prime, so every non-zero residue modulo 23 appears in an exponent in the last sum exactly 7 times, and the summand 1 appears 22 times. Because the sum of the 23rd roots of unity is zero, our answer is $\frac{1}{3}+(22-7) = \frac{46}{3}$.

Second solution: For an alternate approach, we first prove the following identity for an arbitrary complex number a:

$$\sum_{k=0}^{22} \frac{1}{a-z^k} = \frac{23a^{22}}{a^{23}-1}.$$

To see this, let $f(x) = x^{23}-1 = (x-1)(x-z)(x-z^2)\dots(x-z^{22})$. Note that the sum in question is merely $\frac{f'(a)}{f(a)}$, from which the identity follows.

Now, returning to our orignal sum, let $\omega \neq 1$ satisfy $\omega^3=1$. Then

$$\begin{aligned}
\sum_{k=0}^{22} \frac{1}{1+z^k+z^{2k}} &= \frac{1}{\omega^2-\omega}\sum_{k=0}^{22} \frac{1}{\omega-z^k} - \frac{1}{\omega^2-z^k} \\
&= \frac{1}{\omega^2-\omega}\left(\sum_{k=0}^{22} \frac{1}{\omega-z^k} - \sum_{k=0}^{22} \frac{1}{\omega^2-z^k}\right) \\
&= \frac{1}{\omega^2-\omega}\left(\frac{23\omega^{22}}{\omega^{23}-1} - \frac{23\omega^{44}}{\omega^{46}-1}\right) \\
&= \frac{23}{\omega^2-\omega}\left(\frac{\omega}{\omega^2-1} - \frac{\omega^2}{\omega-1}\right) \\
&= \frac{23}{\omega^2-\omega}\,\frac{(\omega^2-\omega)-(\omega-\omega^2)}{2-\omega-\omega^2} \\
&= \frac{46}{3}.
\end{aligned}$$

10. Let N be a positive integer whose decimal representation contains 11235 as a contiguous substring, and let k be a positive integer such that $10^k > N$. Find the minimum possible value of

$$\frac{10^k - 1}{\gcd(N, 10^k - 1)}.$$

Answer: $\boxed{89}$ Set $m = \frac{10^k-1}{\gcd(N,10^k-1)}$. Then, in lowest terms, $\frac{N}{10^k-1} = \frac{a}{m}$ for some integer a. On the other hand, the decimal expansion of $\frac{N}{10^k-1}$ simply consists of the decimal expansion of N, possibly with some padded zeros, repeating. Since N contains 11235 as a contiguous substring, the decimal representation of $\frac{a}{m}$ must as well.

Conversely, if m is relatively prime to 10 and if there exists an a such that the decimal representation of $\frac{a}{m}$ contains the substring 11235, we claim that m is an attainable value for $\frac{10^k-1}{\gcd(N,10^k-1)}$. To see this, note that since m is relatively prime to 10, there exists a value of k such that m divides $10^k - 1$ (for example, $k = \phi(m)$). Letting $ms = 10^k - 1$ and $N = as$, it follows that $\frac{a}{m} = \frac{as}{ms} = \frac{N}{10^k-1}$. Since the decimal expansion of this fraction contains the substring 11235, it follows that N must also, and therefore m is an attainable value.

We are therefore looking for a fraction $\frac{a}{m}$ which contains the substring 11235 in its decimal expansion. Since 1, 1, 2, 3, and 5 are the first five Fibonacci numbers, it makes sense to look at the value of the infinite series

$$\sum_{i=1}^{\infty} \frac{F_i}{10^i}.$$

A simple generating function argument shows that $\sum_{i=1}^{\infty} F_i x^i = \frac{x}{1-x-x^2}$, so substituting $x = 1/10$ leads us to the fraction 10/89 (which indeed begins $0.11235\ldots$).

How do we know no smaller values of m are possible? Well, if a'/m' contains the substring 11235 somewhere in its infinitely repeating decimal expansion, then note that there is an i such that the decimal expansion of the fractional part of $10^i(a'/m')$ begins with $0.11235\ldots$. We can therefore, without loss of generality, assume that the decimal representation of a'/m' begins $0.11235\ldots$. But since the decimal representation of 10/89 begins $0.11235\ldots$, it follows that

$$\left| \frac{10}{89} - \frac{a'}{m'} \right| \leq 10^{-5}.$$

On the other hand, this absolute difference, if non-zero, is at least $\frac{1}{89m'}$. If $m' < 89$, this is at least $\frac{1}{89^2} > 10^{-5}$, and therefore no smaller values of m' are possible.