THE HARVARD-MIT MATHEMATICS TOURNAMENT

COMBINATORICS TEST

This test consists of 10 short-answer problems to be solved individually in 50 minutes. Problems will be weighted with point values after the contest based on how many competitors solve each problem. There is no penalty for guessing.

No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.

Answers should be simplified as much as is reasonably possible and must be exact unless otherwise specified. Rational numbers should be written in lowest terms, although denominators of irrationals need not be rationalized. An nth root should be simplified so that the radicand is not divisible by the nth power of any prime.

Correct mathematical notation must be used. No partial credit will be given unless otherwise specified.

If you believe the test contains an error, please submit your protest in writing to Lobby 10 during lunchtime.

Enjoy!

HMMT 2013

Saturday 16 February 2013

Combinatorics Test

1. A standard 52-card deck contains cards of 4 suits and 13 numbers, with exactly one card for each pairing of suit and number. If Maya draws two cards with replacement from this deck, what is the probability that the two cards have the same suit or have the same number, but not both?

2. If Alex does not sing on Saturday, then she has a 70% chance of singing on Sunday; however, to rest her voice, she never sings on both days. If Alex has a 50% chance of singing on Sunday, find the probability that she sings on Saturday.

3. On a game show, Merble will be presented with a series of 2013 marbles, each of which is either red or blue on the outside. Each time he sees a marble, he can either keep it or pass, but cannot return to a previous marble; he receives 3 points for keeping a red marble, loses 2 points for keeping a blue marble, and gains 0 points for passing. All distributions of colors are equally likely and Merble can only see the color of his current marble. If his goal is to end with exactly one point and he plays optimally, what is the probability that he fails?

4. How many orderings $(a_1, \ldots, a_8)$ of $(1, 2, \ldots, 8)$ exist such that $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 - a_8 = 0$?

5. At a certain chocolate company, each bar is 1 unit long. To make the bars more interesting, the company has decided to combine dark and white chocolate pieces. The process starts with two bars, one completely dark and one completely white. At each step of the process, a new number p is chosen uniformly at random between 0 and 1. Each of the two bars is cut p units from the left, and the pieces on the left are switched: each is grafted onto the opposite bar where the other piece of length p was previously attached. For example, the bars might look like this after the first step:

   Each step after the first operates on the bars resulting from the previous step. After a total of 100 steps, what is the probability that on each bar, the chocolate 1/3 units from the left is the same type of chocolate as that 2/3 units from the left?

6. Values $a_1, \ldots, a_{2013}$ are chosen independently and at random from the set $\{1, \ldots, 2013\}$. What is expected number of distinct values in the set $\{a_1, \ldots, a_{2013}\}$?

7. A single-elimination ping-pong tournament has 2^{2013} players, seeded in order of ability. If the player with seed x plays the player with seed y, then it is possible for x to win if and only if $x \leq y + 3$. For how many players P it is possible for P to win? (In each round of a single elimination tournament, the remaining players are randomly paired up; each player plays against the other player in his pair, with the winner from each pair progressing to the next round and the loser eliminated. This is repeated until there is only one player remaining.)

8. It is known that exactly one of the three (distinguishable) musketeers stole the truffles. Each musketeer makes one statement, in which he either claims that one of the three is guilty, or claims that one of the three is innocent. It is possible for two or more of the musketeers to make the same statement. After hearing their claims, and knowing that exactly one musketeer lied, the inspector is able to deduce who stole the truffles. How many ordered triplets of statements could have been made?

9. Given a permutation σ of $\{1, 2, \ldots, 2013\}$, let $f(\sigma)$ to be the number of fixed points of σ – that is, the number of $k \in \{1, 2, \ldots, 2013\}$ such that $\sigma(k) = k$. If S is the set of all possible permutations σ, compute

$$\sum_{\sigma \in S} f(\sigma)^4.$$

(Here, a *permutation* σ is a bijective mapping from $\{1, 2, \ldots, 2013\}$ to $\{1, 2, \ldots, 2013\}$.)

10. Rosencrantz and Guildenstern each start with \$2013 and are flipping a fair coin. When the coin comes up heads Rosencrantz pays Guildenstern \$1 and when the coin comes up tails Guildenstern pays Rosencrantz \$1. Let $f(n)$ be the number of dollars Rosencrantz is ahead of his starting amount after n flips. Compute the expected value of $\max\{f(0), f(1), f(2), \ldots, f(2013)\}$.

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PUT LABEL HERE

Name ______________________________ Team ID# ________

Organization ______________________ Team ________________

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Score: __________