HMMT 2013

Saturday 16 February 2013

Combinatorics Test

1. A standard 52-card deck contains cards of 4 suits and 13 numbers, with exactly one card for each pairing of suit and number. If Maya draws two cards with replacement from this deck, what is the probability that the two cards have the same suit or have the same number, but not both?

   Answer: $\boxed{\frac{15}{52}}$ After drawing the first card, there are 12 other cards from the same suit and 3 other cards with the same number, so the probability is $\frac{12+3}{52}$.

2. If Alex does not sing on Saturday, then she has a 70% chance of singing on Sunday; however, to rest her voice, she never sings on both days. If Alex has a 50% chance of singing on Sunday, find the probability that she sings on Saturday.

   Answer: $\boxed{\frac{2}{7}}$ Let p be the probability that Alex sings on Saturday. Then, the probability that she sings on Sunday is $.7(1-p)$; setting this equal to .5 gives $p = \frac{2}{7}$.

3. On a game show, Merble will be presented with a series of 2013 marbles, each of which is either red or blue on the outside. Each time he sees a marble, he can either keep it or pass, but cannot return to a previous marble; he receives 3 points for keeping a red marble, loses 2 points for keeping a blue marble, and gains 0 points for passing. All distributions of colors are equally likely and Merble can only see the color of his current marble. If his goal is to end with exactly one point and he plays optimally, what is the probability that he fails?

   Answer: $\boxed{\frac{1}{2^{2012}}}$ First, we note that if all the marbles are red or all are blue, then it is impossible for Merble to win; we claim that he can guarantee himself a win in every other case. In particular, his strategy should be to keep the first red and first blue marble that he encounters, and to ignore all the others. Consequently, the probability that he cannot win is $\frac{2}{2^{2013}} = \frac{1}{2^{2012}}$.

4. How many orderings $(a_1, \ldots, a_8)$ of $(1, 2, \ldots, 8)$ exist such that $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 - a_8 = 0$?

   Answer: $\boxed{4608}$ We can divide the numbers up based on whether they have a + or - before them. Both the numbers following +'s and -'s must add up to 18. Without loss of generality, we can assume the +'s contain the number 1 (and add a factor of 2 at the end to account for this). The possible 4-element sets containing a 1 which add to 18 are $\{1,2,7,8\}, \{1,3,6,8\}, \{1,4,5,8\}, \{1,4,6,7\}$. Additionally, there are 4! ways to order the numbers following a + and 4! ways to order the numbers following a $-$. Thus the total number of possibilities is $4 \times 2 \times 4! \times 4! = 4608$.

5. At a certain chocolate company, each bar is 1 unit long. To make the bars more interesting, the company has decided to combine dark and white chocolate pieces. The process starts with two bars, one completely dark and one completely white. At each step of the process, a new number p is chosen uniformly at random between 0 and 1. Each of the two bars is cut p units from the left, and the pieces on the left are switched: each is grafted onto the opposite bar where the other piece of length p was previously attached. For example, the bars might look like this after the first step:

Each step after the first operates on the bars resulting from the previous step. After a total of 100 steps, what is the probability that on each bar, the chocolate 1/3 units from the left is the same type of chocolate as that 2/3 units from the left?

Answer: $\boxed{\frac{1}{2}\left[\left(\frac{1}{3}\right)^{100}+1\right]}$ If the values of p chosen are $p_1, \dots, p_{100}$, then note that the color of a bar changes at each value of p_i. Consequently, we want to find the probability that exactly an even number of p_i are in $(\frac{1}{3}, \frac{2}{3})$. Summing, this is equal to

$$\binom{100}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{100}+\binom{100}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^{98}+\cdots\binom{100}{100}\left(\frac{1}{3}\right)^{100}\left(\frac{2}{3}\right)^0.$$

To compute, we note that this is equal to

$$\frac{1}{2}\left[\left(\frac{2}{3}+\frac{1}{3}\right)^{100}+\left(\frac{2}{3}-\frac{1}{3}\right)^{100}\right]$$

after expanding using the binomial theorem, since any terms with odd exponents are cancelled out between the two terms.

6. Values $a_1, \dots, a_{2013}$ are chosen independently and at random from the set $\{1, \dots, 2013\}$. What is expected number of distinct values in the set $\{a_1, \dots, a_{2013}\}$?

Answer: $\boxed{\frac{2013^{2013}-2012^{2013}}{2013^{2012}}}$ For each $n \in \{1, 2, \dots, 2013\}$, let $X_n = 1$ if n appears in $\{a_1, a_2, \dots, a_{2013}\}$ and 0 otherwise. Defined this way, $\mathrm{E}[X_n]$ is the probability that n appears in $\{a_1, a_2, \dots, a_{2013}\}$. Since each a_i $(1 \le i \le 2013)$ is not n with probability 2012/2013, the probability that n is none of the a_i's is $\left(\frac{2012}{2013}\right)^{2013}$, so $\mathrm{E}[X_n]$, the probability that n is one of the a_i's, is $1-\left(\frac{2012}{2013}\right)^{2013}$. The expected number of distinct values in $\{a_1, \dots, a_{2013}\}$ is the expected number of $n \in \{1, 2, \dots, 2013\}$ such that $X_n = 1$, that is, the expected value of $X_1 + X_2 + \cdots + X_{2013}$. By linearity of expectation, $\mathrm{E}[X_1 + X_2 + \cdots + X_{2013}] = \mathrm{E}[X_1] + \mathrm{E}[X_2] + \cdots + \mathrm{E}[X_n] = 2013\left(1-\left(\frac{2012}{2013}\right)^{2013}\right) = \frac{2013^{2013}-2012^{2013}}{2013^{2012}}$.

7. A single-elimination ping-pong tournament has 2^{2013} players, seeded in order of ability. If the player with seed x plays the player with seed y, then it is possible for x to win if and only if $x \le y + 3$. For how many players P it is possible for P to win? (In each round of a single elimination tournament, the remaining players are randomly paired up; each player plays against the other player in his pair, with the winner from each pair progressing to the next round and the loser eliminated. This is repeated until there is only one player remaining.)

Answer: $\boxed{6038}$ We calculate the highest seed n that can win. Below, we say that a player x *vicariously* defeats a player y if x defeats y directly or indirectly through some chain (i.e. x defeats x_1, who defeated x_2, …, who defeated x_n, who defeated y for some players $x_1, \dots, x_n$).

We first consider the highest seeds that are capable of making the semifinals. The eventual winner must be able to beat two of these players and thus must be able to beat the second best player in the semifinals. The seed of the player who vicariously beats the 1-seed is maximized if 1 loses to 4 in the first round, 4 to 7 in the second round, etc. Therefore $3 \cdot 2011 + 1 = 6034$ is the maximum value of the highest seed in the semifinals. If 1, and 2 are in different quarters of the draw, then by a similar argument 6035 is the largest possible value of the second best player in the semis, and thus 6038 is the highest that can win. If 1 and 2 are in the same quarter, then in one round the highest remaining seed will not be able to go up by 3, when the player who has vicariously beaten 1 plays the player who vicariously beat 2, so $3 \cdot 2011 - 1 = 6032$ is the highest player the semifinalist from that quarter could be. But then the eventual winner still must be seeded at most 6 above this player, and thus 6038 is still the upper bound.

Therefore 6038 is the worst seed that could possibly win, and can do so if 6034, 6035, 6036, 6038 all make the semis, which is possible (it is not difficult to construct such a tournament). Then, note that any player x with a lower seed can also win for some tournament – in particular, it suffices to take the tournament where it is possible for player 6038 to win and switch the positions of 6038 and x. Consequently, there are 6038 players for whom it is possible to win under some tournament.

8. It is known that exactly one of the three (distinguishable) musketeers stole the truffles. Each musketeer makes one statement, in which he either claims that one of the three is guilty, or claims that one of the three is innocent. It is possible for two or more of the musketeers to make the same statement. After hearing their claims, and knowing that exactly one musketeer lied, the inspector is able to deduce who stole the truffles. How many ordered triplets of statements could have been made?

Answer: $\boxed{99}$ We divide into cases, based on the number of distinct people that statements are made about.

- The statements are made about 3 distinct people. Then, since exactly one person is guilty, and because exactly one of the three lied, there are either zero statements of guilt or two statements of guilt possible; in either case, it is impossible by symmetry to determine who is guilty or innocent.
- The statements are made about 2 distinct people or 1 distinct person. Then, either at least two of the statements are the same, or all are different.
  - If two statements are the same, then those two statements must be true because only one musketeer lied. Consequently, the lone statement must be false. If all the statements are about the same person, there there must be 2 guilty claims and 1 innocence claim (otherwise we would not know which of the other two people was guilty). Then, there are 3 choices for who the statement is about and 3 choices for who makes the innocence claim, for a $3 \cdot 3 = 9$ triplets of statements. Meanwhile, if the statements are about two different people, then this is doable unless both of the distinct statements imply guilt for the person concerned (i.e. where there are two guilty accusations against one person and one claim of innocence against another). Consequently, there are 3 sets of statements that can be made, $3 \cdot 2 = 6$ ways to determine who they are made about, and 3 ways to determine who makes which statement, for a total of $3 \cdot 6 \cdot 3 = 54$ triplets in this case.
  - If all the statements are different, then they must be about two different people. Here, there must be one person, who we will call A, who has both a claim of innocence and an accusation of guilt against him. The last statement must concern another person, B. If the statement accuses B of being guilty, then we can deduce that he is the guilty one. On the other hand, if the statement claims that B is innocent, either of the other two musketeers could be guilty. Consequently, there are $3 \cdot 2 = 6$ ways to choose A and B, and $3! = 6$ ways to choose who makes which statement, for a total of $6 \cdot 6 = 36$ triplets of statements.

In total, we have $9 + 54 + 36 = 99$ possible triplets of statements.

9. Given a permutation σ of $\{1, 2, \dots, 2013\}$, let $f(\sigma)$ to be the number of fixed points of σ – that is, the number of $k \in \{1, 2, \dots, 2013\}$ such that $\sigma(k) = k$. If S is the set of all possible permutations σ, compute

$$\sum_{\sigma \in S} f(\sigma)^4.$$

(Here, a *permutation* σ is a bijective mapping from $\{1, 2, \dots, 2013\}$ to $\{1, 2, \dots, 2013\}$.)

Answer: $\boxed{15(2013!)}$ First, note that

$$\sum_{\sigma \in S} f(\sigma)^4 = \sum_{\sigma \in S} \sum_{1 \le a_1, a_2, a_3, a_4 \le 2013} g(\sigma, a_1, a_2, a_3, a_4),$$

where $g(\sigma, a_1, a_2, a_3, a_4) = 1$ if all a_i are fixed points of σ and 0 otherwise. (The a_i's need not be distinct.) Switching the order of summation, we find that the desired sum is

$$\sum_{1 \le a_1, a_2, a_3, a_4 \le 2013} \sum_{\sigma \in S} g(\sigma, a_1, a_2, a_3, a_4).$$

Note that the inner sum is equal to the number of permutations on $\{1, 2, \ldots, 2013\}$ that fix a_1, a_2, a_3, and a_4. This depends on the number of distinct values the a_is take. If they take on exactly k distinct values, then the inner sum will evaluate to $(2013-k)!$, because σ can be any permutation of the remaining $2013-k$ elements. (For example, if $a_1 = a_2$ but a_1, a_3, and a_4 are distinct, the inner sum is 2010! because σ can be any permutation that fixes a_1, a_3, and a_4.)

Now, suppose we are given which of the a_i are equal (for example, we could be given $a_1 = a_2$ but a_1, a_3, a_4 mutually distinct, as per the above example). Assuming there are k distinct values among the a_i, there are $2013(2013-1)\cdots(2013-k+1)$ ways to choose the a_i. At this point, there are $(2013-k)!$ ways to choose σ on the remaining $(2013-k)$ values such that it fixes the a_i, for a total of 2013! choices for $(\sigma, a_1, a_2, a_3, a_4)$ such that $g(\sigma, a_1, a_2, a_3, a_4) = 1$ and the a_i satisfy the correct equality relations.

Thus the answer is 2013! times the number of ways to choose equivalence classes on the a_i, so the problem reduces to finding the number of ways to partition 4 elements into nonempty sets. This process can be accelerated by doing casework based on the number of sets:

(a) One set must contain all four elements, only one possibility. (i.e. all the a_is are equal)

(b) Either one set contains 3 elements and the other contains the fourth (4 possibilities) or one set contains 2 elements and the other contains the other two (3 possibilities). (i.e. there are two distinct values of a_i)

(c) One set contains two elements, the other two each contain one. There are $\binom{4}{2} = 6$ ways to choose the two elements in the set with two elements, and this uniquely determines the partition. (i.e. there are three distinct values of a_i)

(d) All sets contain one element, in which case there is only one possibility. (i.e. all the a_i are distinct)

Thus the number of ways to construct such a partition is $1 + 4 + 3 + 6 + 1 = 15$, and our answer is $15 \cdot 2013!$.

10. Rosencrantz and Guildenstern each start with \$2013 and are flipping a fair coin. When the coin comes up heads Rosencrantz pays Guildenstern \$1 and when the coin comes up tails Guildenstern pays Rosencrantz \$1. Let $f(n)$ be the number of dollars Rosencrantz is ahead of his starting amount after n flips. Compute the expected value of $\max\{f(0), f(1), f(2), \ldots, f(2013)\}$.

Answer: $\boxed{\frac{-1}{2} + \frac{(1007)\binom{2013}{1006}}{2^{2012}}}$ We want to calculate $\Gamma = \sum_{i=0}^{\infty} i \cdot P(\text{max profit}=i)$, where we consider the maximum profit Rosencrantz has at any point over the first 2013 coin flips. By summation by parts this is equal to $\sum_{a=1}^{2013} P(\text{max profit} \geq a)$.

Let p_a be the probability that Rosencrantz' max profit is at least a and let $E_{n,a}$ be the set of sequences of flips such that Rosencrantz first reaches a profit of a on exactly the nth flip. Let Q_a^+, Q_a^-, Q_a be the sets such that after all 2013 flips Rosencrantz' final profit is (respectively) greater than, less than, or equal to a.

Then,

$$\begin{aligned}
\Gamma &= \sum_{a=1}^{2013} P(\text{max profit } \geq a) \\
&= \sum_{a=1}^{2013} \sum_{n=a}^{2013} P(E_{n,a}) \\
&= \sum_{a=1}^{2013} \sum_{n=a}^{2013} P(E_{n,a} \cap Q_a^+) + P(E_{n,a} \cap Q_a^-) + P(E_{n,a} \cap Q_a).
\end{aligned}$$

By symmetry, $P(E_{n,a} \cap Q_a^+) = P(E_{n,a} \cap Q_a^-)$ because, for any sequence in $E_{n,a} \cap Q_a^+$, we can reverse all the flips after the nth flip to get a sequence in $E_{n,a} \cap Q_a^-$, and vice-versa.

Furthermore, $\sum_{n=a}^{2013} P(E_{n,a} \cap Q_a^+) = P(Q_a^+)$ and $\sum_{n=a}^{2013} P(E_{n,a} \cap Q_a) = P(Q_a)$.

So we have

$$\sum_{a=1}^{2013} P(\text{max profit } {\geq} a) = \sum_{a=1}^{2013} \left(P(Q_a) + 2P(Q_a^+)\right).$$

Since by symmetry $P(Q_a) = P(Q_{-a})$ and we have an odd number of flips, we have $\sum_{a=1}^{2013} P(Q_a) = \frac{1}{2}$.

Also $P(Q_a^+) = \frac{1}{2^{2013}} \sum_{k=\lceil \frac{2014+a}{2} \rceil}^{2013} \binom{2013}{k}$.

So the rest is just computation. We have:

$$\begin{aligned}
\Gamma &= \frac{1}{2} + \frac{1}{2^{2012}} \sum_{a=1}^{2013} \sum_{k=\lceil \frac{2014+a}{2} \rceil}^{2013} \binom{2013}{k} \\
&= \frac{1}{2} + \frac{1}{2^{2012}} \sum_{k=1008}^{2013} \sum_{a=1}^{2k-2014} \binom{2013}{k} \\
&= \frac{1}{2} + \frac{1}{2^{2012}} \sum_{k=1008}^{2013} \binom{2013}{k} (k + k - 2013 - 1) \\
&= \frac{1}{2} + \frac{1}{2^{2012}} \sum_{k=1008}^{2013} 2013\binom{2012}{k-1} - 2013\binom{2012}{k} - \binom{2013}{k} \\
&= \frac{1}{2} + \frac{2013\binom{2012}{1007} - 2^{2012} + \binom{2013}{1007}}{2^{2012}} \\
&= \frac{-1}{2} + \frac{(1007)\binom{2013}{1006}}{2^{2012}}.
\end{aligned}$$

So the answer is $\frac{-1}{2} + \frac{(1007)\binom{2013}{1006}}{2^{2012}}$ (for reference, approximately 35.3).