HMMT 2013

Saturday 16 February 2013

Geometry Test

1. Jarris the triangle is playing in the (x, y) plane. Let his maximum y coordinate be k. Given that he has side lengths 6, 8, and 10 and that no part of him is below the x-axis, find the minimum possible value of k.

   Answer: $\boxed{\frac{24}{5}}$ By playing around, we find that Jarris should have his hypotenuse flat on the x-axis. The desired minimum value of k is then the length of the altitude to the hypotenuse. Thus, by computing the area of the triangle in two ways, $\frac{1}{2} \cdot 10 \cdot k = \frac{1}{2} \cdot 6 \cdot 8$ and so $k = \frac{24}{5}$.

2. Let $ABCD$ be an isosceles trapezoid such that $AD = BC$, $AB = 3$, and $CD = 8$. Let E be a point in the plane such that $BC = EC$ and $AE \perp EC$. Compute AE.

   Answer: $\boxed{2\sqrt{6}}$ Let $r = BC = EC = AD$. $\triangle ACE$ has right angle at E, so by the Pythagorean Theorem,

$$AE^2 = AC^2 - CE^2 = AC^2 - r^2$$

   Let the height of $\triangle ACD$ at A intersect DC at F. Once again, by the Pythagorean Theorem,

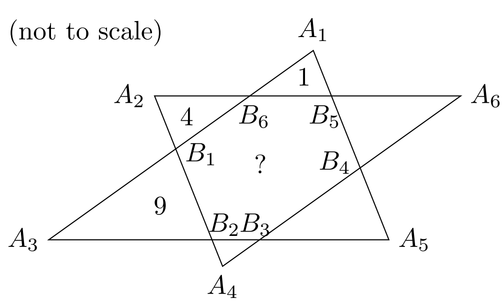
$$AC^2 = FC^2 + AF^2 = \left(\frac{8-3}{2} + 3\right)^2 + AD^2 - DF^2 = \left(\frac{11}{2}\right)^2 + r^2 - \left(\frac{5}{2}\right)^2$$

   Plugging into the first equation,

$$AE^2 = \left(\frac{11}{2}\right)^2 + r^2 - \left(\frac{5}{2}\right)^2 - r^2,$$

   so $AE = 2\sqrt{6}$.

3. Let $A_1A_2A_3A_4A_5A_6$ be a convex hexagon such that $A_iA_{i+2} \parallel A_{i+3}A_{i+5}$ for $i = 1, 2, 3$ (we take $A_{i+6} = A_i$ for each i). Segment A_iA_{i+2} intersects segment $A_{i+1}A_{i+3}$ at B_i, for $1 \le i \le 6$, as shown. Furthermore, suppose that $\triangle A_1A_3A_5 \cong \triangle A_4A_6A_2$. Given that $[A_1B_5B_6] = 1$, $[A_2B_6B_1] = 4$, and $[A_3B_1B_2] = 9$ (by $[XYZ]$ we mean the area of $\triangle XYZ$), determine the area of hexagon $B_1B_2B_3B_4B_5B_6$.



   Answer: $\boxed{22}$ Because $B_6A_3B_3A_6$ and $B_1A_4B_4A_1$ are parallelograms, $B_6A_3 = A_6B_3$ and $A_1B_1 = A_4B_4$. By the congruence of the large triangles $A_1A_3A_5$ and $A_2A_4A_6$, $A_1A_3 = A_4A_6$. Thus, $B_6A_3 + A_1B_1 - A_1A_3 = A_6B_3 + A_4B_4 - A_4A_6$, so $B_6B_1 = B_3B_4$. Similarly, opposite sides of hexagon $B_1B_2B_3B_4B_5B_6$ are equal, and implying that the triangles opposite each other on the outside of this hexagon are congruent.

   Furthermore, by definition $B_5B_6 \parallel A_3A_5, B_3B_4 \parallel A_1A_3, B_6B_1 \parallel A_4A_6$ and$B_1B_2 \parallel A_1A_5$. Let the area of triangle $A_1A_3A_5$ and triangle $A_2A_4A_6$ be k^2. Then, by similar triangles,

$$\sqrt{\frac{1}{k^2}} = \frac{A_1B_6}{A_1A_3}$$
$$\sqrt{\frac{4}{k^2}} = \frac{B_6B_1}{A_4A_6} = \frac{B_1B_6}{A_1A_3}$$
$$\sqrt{\frac{9}{k^2}} = \frac{A_3B_1}{A_1A_3}$$

Summing yields $6/k = 1$, so $k^2 = 36$. To finish, the area of $B_1B_2B_3B_4B_5B_6$ is equivalent to the area of the triangle $A_1A_3A_5$ minus the areas of the smaller triangles provided in the hypothesis. Thus, our answer is $36 - 1 - 4 - 9 = 22$.

4. Let ω_1 and ω_2 be circles with centers O_1 and O_2, respectively, and radii r_1 and r_2, respectively. Suppose that O_2 is on ω_1. Let A be one of the intersections of ω_1 and ω_2, and B be one of the two intersections of line O_1O_2 with ω_2. If $AB = O_1A$, find all possible values of $\frac{r_1}{r_2}$.

   Answer: $\boxed{\frac{-1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}}$ There are two configurations to this problem, namely, B in between the segment O_1O_2 and B on the ray O_1O_2 passing through the side of O_2 Case 1: Let us only consider the triangle ABO_2. $AB = AO_1 = O_1O_2 = r_1$ because of the hypothesis and AO_1 and O_1O_2 are radii of w_1. $O_2B = O_2A = r_2$ because they are both radii of w_2.

   Then by the isosceles triangles, $\angle AO_1B = \angle ABO_1 = \angle ABO_2 = \angle O_2AB$. Thus can establish that $\triangle ABO_1 \sim \triangle O_2AB$.

   Thus,
$$\frac{r_2}{r_1} = \frac{r_1}{r_2 - r_1}$$
$$r_1^2 - r_2^2 + r_1r_2 = 0$$

   By straightforward quadratic equation computation and discarding the negative solution,
$$\frac{r_1}{r_2} = \frac{-1+\sqrt{5}}{2}$$

   Case 2: Similar to case 1, let us only consider the triangle ABO_1. $AB = AO_1 = O_1O_2 = r_1$ because of the hypothesis and AO_1 and O_1O_2 are radii of w_1. $O_2B = O_2A = r_2$ because they are both radii of w_2.

   Then by the isosceles triangles, $\angle AO_1B = \angle ABO_1 = \angle ABO_2 = \angle O_2AB$. Thus can establish that $\triangle ABO_1 \sim \triangle O_2AB$.

   Now,
$$\frac{r_2}{r_1} = \frac{r_1}{r_2 + r_1}$$
$$r_1^2 - r_2^2 - r_1r_2 = 0$$

   By straightforward quadratic equation computation and discarding the negative solution,
$$\frac{r_1}{r_2} = \frac{1+\sqrt{5}}{2}$$

5. In triangle ABC, $\angle A = 45°$ and M is the midpoint of $\overline{BC}$. $\overline{AM}$ intersects the circumcircle of ABC for the second time at D, and $AM = 2MD$. Find $\cos\angle AOD$, where O is the circumcenter of ABC.

   Answer: $\boxed{-\frac{1}{8}}$ $\angle BAC = 45°$, so $\angle BOC = 90°$. If the radius of the circumcircle is r, $BC = \sqrt{2}r$, and $BM = CM = \frac{\sqrt{2}}{2}r$. By power of a point, $BM \cdot CM = AM \cdot DM$, so $AM = r$ and $DM = \frac{1}{2}r$, and $AD = \frac{3}{2}r$. Using the law of cosines on triangle AOD gives $\cos\angle AOD = -\frac{1}{8}$.

6. Let $ABCD$ be a quadrilateral such that $\angle ABC = \angle CDA = 90^\circ$, and $BC = 7$. Let E and F be on BD such that AE and CF are perpendicular to BD. Suppose that $BE = 3$. Determine the product of the smallest and largest possible lengths of DF.

Answer: $\boxed{9}$ By inscribed angles, $\angle CDB = \angle CAB$, and $\angle ABD = \angle ACD$. By definition, $\angle AEB = \angle CDA = \angle ABC = \angle CFA$. Thus, $\triangle ABE \sim \triangle ADC$ and $\triangle CDF \sim \triangle CAB$. This shows that

$$\frac{BE}{AB} = \frac{CD}{CA} \text{ and } \frac{DF}{CD} = \frac{AB}{BD}$$

Based on the previous two equations, it is sufficient to conclude that $3 = EB = FD$. Thus, FD must equal to 3, and the product of its largest and smallest length is 9.

7. Let ABC be an obtuse triangle with circumcenter O such that $\angle ABC = 15^\circ$ and $\angle BAC > 90^\circ$. Suppose that AO meets BC at D, and that $OD^2 + OC \cdot DC = OC^2$. Find $\angle C$.

Answer: $\boxed{35}$ Let the radius of the circumcircle of $\triangle ABC$ be r.

$$OD^2 + OC \cdot CD = OC^2$$
$$OC \cdot CD = OC^2 - OD^2$$
$$OC \cdot CD = (OC + OD)(OC - OD)$$
$$OC \cdot CD = (r + OD)(r - OD)$$

By the power of the point at D,

$$OC \cdot CD = BD \cdot DC$$
$$r = BD$$

Then, $\triangle OBD$ and $\triangle OAB$ and $\triangle AOC$ are isosceles triangles. Let $\angle DOB = \alpha$. $\angle BAO = 90 - \frac{\alpha}{2}$. In $\triangle ABD$, $15 + 90 - \frac{\alpha}{2} = \alpha$. This means that $\alpha = 70$. Furthermore, $\angle ACB$ intercepts minor arc AB, thus $\angle ACB = \frac{\angle AOB}{2} = \frac{70}{2} = 35$

8. Let $ABCD$ be a convex quadrilateral. Extend line CD past D to meet line AB at P and extend line CB past B to meet line AD at Q. Suppose that line AC bisects $\angle BAD$. If $AD = \frac{7}{4}$, $AP = \frac{21}{2}$, and $AB = \frac{14}{11}$, compute AQ.

Answer: $\boxed{\frac{42}{13}}$ We prove the more general statement $\frac{1}{AB} + \frac{1}{AP} = \frac{1}{AD} + \frac{1}{AQ}$, from which the answer easily follows.

Denote $\angle BAC = \angle CAD = \gamma$, $\angle BCA = \alpha$, $\angle ACD = \beta$. Then we have that by the law of sines, $\frac{AC}{AB} + \frac{AC}{AP} = \frac{\sin(\gamma+\alpha)}{\sin(\alpha)} + \frac{\sin(\gamma-\beta)}{\sin(\beta)} = \frac{\sin(\gamma-\alpha)}{\sin(\alpha)} + \frac{\sin(\gamma+\beta)}{\sin(\beta)} = \frac{AC}{AD} + \frac{AC}{AQ}$ where we have simply used the sine addition formula for the middle step.

Dividing the whole equation by AC gives the desired formula, from which we compute $AQ = (\frac{11}{14} + \frac{2}{21} - \frac{4}{7})^{-1} = \frac{42}{13}$.

9. Pentagon $ABCDE$ is given with the following conditions:

(a) $\angle CBD + \angle DAE = \angle BAD = 45^\circ$, $\angle BCD + \angle DEA = 300^\circ$

(b) $\frac{BA}{DA} = \frac{2\sqrt{2}}{3}$, $CD = \frac{7\sqrt{5}}{3}$, and $DE = \frac{15\sqrt{2}}{4}$

(c) $AD^2 \cdot BC = AB \cdot AE \cdot BD$

Compute BD.

Answer: $\boxed{\sqrt{39}}$ As a preliminary, we may compute that by the law of cosines, the ratio $\frac{AD}{BD} = \frac{3}{\sqrt{5}}$.

Now, construct the point P in triangle ABD such that $\triangle APB \sim \triangle AED$. Observe that $\frac{AP}{AD} = \frac{AE \cdot AB}{AD \cdot AD} = \frac{BC}{BD}$ (where we have used first the similarity and then condition 3). Furthermore, $\angle CBD = \angle DAB - \angle DAE = \angle DAB - \angle PAB = \angle PAD$ so by SAS, we have that $\triangle CBD \sim \triangle PAD$.

Therefore, by the similar triangles, we may compute $PB = DE \cdot \frac{AB}{AD} = 5$ and $PD = CD \cdot \frac{AD}{BD} = 7$. Furthermore, $\angle BPD = 360 - \angle BPA - \angle DPA = 360 - \angle BCD - \angle DEA = 60$ and therefore, by the law of cosines, we have that $BD = \sqrt{39}$.

10. Triangle ABC is inscribed in a circle ω. Let the bisector of angle A meet ω at D and BC at E. Let the reflections of A across D and C be D' and C', respectively. Suppose that $\angle A = 60°$, $AB = 3$, and $AE = 4$. If the tangent to ω at A meets line BC at P, and the circumcircle of APD' meets line BC at F (other than P), compute FC'.

 Answer: $\boxed{2\sqrt{13 - 6\sqrt{3}}}$ First observe that by angle chasing, $\angle PAE = 180 - \frac{1}{2}\angle BAC - \angle ABC = \angle AEP$, so by the cyclic quadrilateral $APD'F$, $\angle EFD' = \angle PAE = \angle PEA = \angle D'EF$. Thus, $ED'F$ is isosceles.

 Define B' to be the reflection of A about B, and observe that $B'C'||EF$ and $B'D'C'$ is isosceles. It follows that $B'EFC'$ is an isosceles trapezoid, so $FC' = B'E$, which by the law of cosines, is equal to $\sqrt{AB'^2 + AE^2 - 2AB \cdot AE \cos 30} = 2\sqrt{13 - 6\sqrt{3}}$.