HMMT 2013

Saturday 16 February 2013

Guts Round

1. [**4**] Arpon chooses a positive real number k. For each positive integer n, he places a marker at the point (n, nk) in the (x, y) plane. Suppose that two markers whose x coordinates differ by 4 have distance 31. What is the distance between the markers at $(7, 7k)$ and $(19, 19k)$?

   Answer: $\boxed{93}$ The difference of the x-coordinates of the markers is $12 = 3 \cdot 4$. Thus, by similar triangles (where we draw right triangles whose legs are parallel to the axes and whose hypotenuses lie along the line $y = kx$), the distance between the markers is $3 \cdot 31 = 93$.

2. [**4**] The real numbers x, y, z satisfy $0 \leq x \leq y \leq z \leq 4$. If their squares form an arithmetic progression with common difference 2, determine the minimum possible value of $|x - y| + |y - z|$.

   Answer: $\boxed{4 - 2\sqrt{3}}$ Clearly $|x-y|+|y-z| = z - x = \frac{z^2-x^2}{z+x} = \frac{4}{z+x}$, which is minimized when $z = 4$ and $x = \sqrt{12}$. Thus, our answer is $4 - \sqrt{12} = 4 - 2\sqrt{3}$.

3. [**4**] Find the rightmost non-zero digit of the expansion of $(20)(13!)$.

   Answer: $\boxed{6}$ We can rewrite this as $(10 * 2)(13 * 12 * 11 * 10 * 9 * 8 * 7 * 6 * 5 * 4 * 3 * 2 * 1) = (10^3)(2 * 13 * 12 * 11 * 9 * 8 * 7 * 6 * 4 * 3)$; multiplying together the units digits for the terms not equal to 10 reveals that the rightmost non-zero digit is 6.

4. [**4**] Spencer is making burritos, each of which consists of one wrap and one filling. He has enough filling for up to four beef burritos and three chicken burritos. However, he only has five wraps for the burritos; in how many orders can he make exactly five burritos?

   Answer: $\boxed{25}$ Spencer's burrito-making can include either 3, 2, or 1 chicken burrito; consequently, he has $\binom{5}{3} + \binom{5}{2} + \binom{5}{1} = 25$ orders in which he can make burritos.

5. [**5**] Rahul has ten cards face-down, which consist of five distinct pairs of matching cards. During each move of his game, Rahul chooses one card to turn face-up, looks at it, and then chooses another to turn face-up and looks at it. If the two face-up cards match, the game ends. If not, Rahul flips both cards face-down and keeps repeating this process. Initially, Rahul doesn't know which cards are which. Assuming that he has perfect memory, find the smallest number of moves after which he can guarantee that the game has ended.

   Answer: $\boxed{4}$ Label the 10 cards $a_1, a_2, ..., a_5, b_1, b_2, ..., b_5$ such that a_i and b_i match for $1 \leq i \leq 5$.

   First, we'll show that Rahul cannot always end the game in less than 4 moves, in particular, when he turns up his fifth card (during the third move), it is possible that the card he flips over is not one which he has yet encountered; consequently, he will not guarantee being able to match it, so he cannot guarantee that the game can end in three moves.

   However, Rahul can always end the game in 4 moves. To do this, he can turn over 6 distinct cards in his first 3 moves. If we consider the 5 sets of cards $\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}, \{a_4, b_4\}, \{a_5, b_5\}$, then by the pigeonhole principle, at least 2 of the 6 revealed cards must be from the same set. Rahul can then turn over those 2 cards on the fourth move, ending the game.

6. [**5**] Let R be the region in the Cartesian plane of points (x, y) satisfying $x \geq 0$, $y \geq 0$, and $x + y + \lfloor x \rfloor + \lfloor y \rfloor \leq 5$. Determine the area of R.

   Answer: $\boxed{\frac{9}{2}}$ We claim that a point in the first quadrant satisfies the desired property if the point is below the line $x + y = 3$ and does not satisfy the desired property if it is above the line.

   To see this, for a point inside the region, $x + y < 3$ and $\lfloor x \rfloor + \lfloor y \rfloor \leq x + y < 3$ However, $\lfloor x \rfloor + \lfloor y \rfloor$ must equal to an integer. Thus, $\lfloor x \rfloor + \lfloor y \rfloor \leq 2$. Adding these two equations, $x + y + \lfloor x \rfloor + \lfloor y \rfloor < 5$, which satisfies the desired property. Conversely, for a point outside the region, $\lfloor x \rfloor + \lfloor y \rfloor + \{x\} + \{y\} = x + y > 3$ However, $\{x\} + \{y\} < 2$. Thus, $\lfloor x \rfloor + \lfloor y \rfloor > 1$, so $\lfloor x \rfloor + \lfloor y \rfloor \geq 2$, implying that $x + y + \lfloor x \rfloor + \lfloor y \rfloor > 5$.

To finish, R is the region bounded by the x-axis, the y-axis, and the line $x+y=3$ is a right triangle whose legs have length 3. Consequently, R has area $\frac{9}{2}$.

7. [**5**] Find the number of positive divisors d of $15! = 15 \cdot 14 \cdot \dots \cdot 2 \cdot 1$ such that $\gcd(d, 60) = 5$.

Answer: $\boxed{36}$ Since $\gcd(d, 60) = 5$, we know that $d = 5^i d'$ for some integer $i > 0$ and some integer d' which is relatively prime to 60. Consequently, d' is a divisor of $(15!)/5$; eliminating common factors with 60 gives that d' is a factor of $(7^2)(11)(13)$, which has $(2+1)(1+1)(1+1) = 12$ factors. Finally, i can be $1, 2$, or 3, so there are a total of $3 \cdot 12 = 36$ possibilities.

8. [**5**] In a game, there are three indistinguishable boxes; one box contains two red balls, one contains two blue balls, and the last contains one ball of each color. To play, Raj first predicts whether he will draw two balls of the same color or two of different colors. Then, he picks a box, draws a ball at random, looks at the color, and replaces the ball in the same box. Finally, he repeats this; however, the boxes are not shuffled between draws, so he can determine whether he wants to draw again from the same box. Raj wins if he predicts correctly; if he plays optimally, what is the probability that he will win?

Answer: $\boxed{\frac{5}{6}}$ Call the box with two red balls box 1, the box with one of each color box 2, and the box with two blue balls box 3. Without loss of generality, assume that the first ball that Bob draws is red. If Bob picked box 1, then he would have picked a red ball with probability 1, and if Bob picked box 2, then he would have picked a red ball with probability $\frac{1}{2}$. Therefore, the probability that he picked box 1 is $\frac{1}{1+\frac{1}{2}} = \frac{2}{3}$, and the probability that he picked box 2 is $\frac{1}{3}$. We will now consider both possible predictions and find which one gives a better probability of winning, assuming optimal play.

If Bob predicts that he will draw two balls of the same color, then there are two possible plays: he draws from the same box, or he draws from a different box. If he draws from the same box, then in the $\frac{2}{3}$ chance that he originally picked box 1, he will always win, and in the $\frac{1}{3}$ chance that he picked box 2, he will win with probability $\frac{1}{2}$, for a total probability of $\frac{2}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{5}{6}$. If he draws from a different box, then if he originally picked box 1, he will win with probability $\frac{1}{4}$ and if he originally picked box 2, he will win with probability $\frac{1}{2}$, for a total probability of $\frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{3}$.

If Bob predicts that he will draw two balls of different colors, then we can consider the same two possible plays. Using similar calculations, if he draws from the same box, then he will win with probability $\frac{1}{6}$, and if he draws from a different box, then he will win with probability $\frac{2}{3}$. Looking at all cases, Bob's best play is to predict that he will draw two balls of the same color and then draw the second ball from the same box, with a winning probability of $\frac{5}{6}$.

9. [**6**] I have 8 unit cubes of different colors, which I want to glue together into a $2 \times 2 \times 2$ cube. How many distinct $2 \times 2 \times 2$ cubes can I make? Rotations of the same cube are not considered distinct, but reflections are.

Answer: $\boxed{1680}$ Our goal is to first pin down the cube, so it can't rotate. Without loss of generality, suppose one of the unit cubes is purple, and let the purple cube be in the top left front position. Now, look at the three positions that share a face with the purple cube. There are $\binom{7}{3}$ ways to pick the three cubes that fill those positions and two ways to position them that are rotationally distinct. Now, we've taken care of any possible rotations, so there are simply 4! ways to position the final four cubes. Thus, our answer is $\binom{7}{3} \cdot 2 \cdot 4! = 1680$ ways.

10. [**6**] Wesyu is a farmer, and she's building a cao (a relative of the cow) pasture. She starts with a triangle $A_0A_1A_2$ where angle A_0 is $90°$, angle A_1 is $60°$, and A_0A_1 is 1. She then extends the pasture. First, she extends A_2A_0 to A_3 such that $A_3A_0 = \frac{1}{2}A_2A_0$ and the new pasture is triangle $A_1A_2A_3$. Next, she extends A_3A_1 to A_4 such that $A_4A_1 = \frac{1}{6}A_3A_1$. She continues, each time extending A_nA_{n-2} to A_{n+1} such that $A_{n+1}A_{n-2} = \frac{1}{2^n-2}A_nA_{n-2}$. What is the smallest K such that her pasture never exceeds an area of K?

Answer: $\boxed{\sqrt{3}}$ First, note that for any i, after performing the operation on triangle $A_iA_{i+1}A_{i+2}$, the resulting pasture is triangle $A_{i+1}A_{i+2}A_{i+3}$. Let K_i be the area of triangle $A_iA_{i+1}A_{i+2}$. From

$A_{n+1}A_{n-2} = \frac{1}{2^n-2}A_nA_{n-2}$ and $A_nA_{n+1} = A_nA_{n-2}+A_{n-2}A_{n+1}$, we have $A_nA_{n+1} = (1+\frac{1}{2^n-2})A_nA_{n-2}$. We also know that the area of a triangle is half the product of its base and height, so if we let the base of triangle $A_{n-2}A_{n-1}A_n$ be A_nA_{n-2}, its area is $K_{n-2} = \frac{1}{2}hA_nA_{n-2}$. The area of triangle $A_{n-1}A_nA_{n+1}$ is $K_{n-1} = \frac{1}{2}hA_nA_{n+1}$. The h's are equal because the distance from A_{n-1} to the base does not change.

We now have $\frac{K_{n-1}}{K_{n-2}} = \frac{A_nA_{n+1}}{A_nA_{n-2}} = 1 + \frac{1}{2^n-2} = \frac{2^n-1}{2^n-2}$. Therefore, $\frac{K_1}{K_0} = \frac{3}{2}$, $\frac{K_2}{K_0} = \frac{K_2}{K_1}\frac{K_1}{K_0} = \frac{7}{6} \cdot \frac{3}{2} = \frac{7}{4}$, $\frac{K_3}{K_0} = \frac{K_3}{K_2}\frac{K_2}{K_0} = \frac{15}{14} \cdot \frac{7}{4} = \frac{15}{8}$. We see the pattern $\frac{K_n}{K_0} = \frac{2^{n+1}-1}{2^n}$, which can be easily proven by induction. As n approaches infinity, $\frac{K_n}{K_0}$ grows arbitrarily close to 2, so the smallest K such that the pasture never exceeds an area of K is $2K_0 = \sqrt{3}$.

11. [**6**] Compute the prime factorization of 1007021035035021007001. (You should write your answer in the form $p_1^{e_1}p_2^{e_2}\ldots p_k^{e_k}$, where $p_1, \ldots, p_k$ are distinct prime numbers and $e_1, \ldots, e_k$ are positive integers.)

Answer: $\boxed{7^7 \cdot 11^7 \cdot 13^7}$ The number in question is

$$\sum_{i=0}^{7} \binom{7}{i} 1000^i = (1000+1)^7 = 1001^7 = 7^7 \cdot 11^7 \cdot 13^7.$$

12. [**6**] For how many integers $1 \leq k \leq 2013$ does the decimal representation of k^k end with a 1?

Answer: $\boxed{202}$ We claim that this is only possible if k has a units digit of 1. Clearly, it is true in these cases. Additionally, k^k cannot have a units digit of 1 when k has a units digit of $2, 4, 5, 6$, or 8. If k has a units digit of 3 or 7, then k^k has a units digit of 1 if and only if $4|k$, a contradiction. Similarly, if k has a units digit of 9, then k^k has a units digit of 1 if and only if $2|k$, also a contradiction. Since there are 202 integers between 1 and 2013, inclusive, with a units digit of 1, there are 202 such k which fulfill our criterion.

13. [**8**] Find the smallest positive integer n such that $\frac{5^{n+1}+2^{n+1}}{5^n+2^n} > 4.99$.

Answer: $\boxed{7}$ Writing $5^{n+1} = 5 \cdot 5^n$ and $2^{n+1} = 2 \cdot 2^n$ and cross-multiplying yields $0.01 \cdot 5^n > 2.99 \cdot 2^n$, and re-arranging yields $(2.5)^n > 299$. A straightforward calculation shows that the smallest n for which this is true is $n = 7$.

14. [**8**] Consider triangle ABC with $\angle A = 2\angle B$. The angle bisectors from A and C intersect at D, and the angle bisector from C intersects $\overline{AB}$ at E. If $\frac{DE}{DC} = \frac{1}{3}$, compute $\frac{AB}{AC}$.

Answer: $\boxed{\frac{7}{9}}$ Let $AE = x$ and $BE = y$. Using angle-bisector theorem on $\triangle ACE$ we have $x : DE = AC : DC$, so $AC = 3x$. Using some angle chasing, it is simple to see that $\angle ADE = \angle AED$, so $AD = AE = x$. Then, note that $\triangle CDA \sim \triangle CEB$, so $y : (DC+DE) = x : DC$, so $y : x = 1+\frac{1}{3} = \frac{4}{3}$, so $AB = x + \frac{4}{3}x = \frac{7}{3}x$. Thus the desired answer is $AB : AC = \frac{7}{3}x : 3x = \frac{7}{9}$.

15. [**8**] Tim and Allen are playing a match of *tenus*. In a match of *tenus*, the two players play a series of games, each of which is won by one of the two players. The match ends when one player has won exactly two more games than the other player, at which point the player who has won more games wins the match. In odd-numbered games, Tim wins with probability 3/4, and in the even-numbered games, Allen wins with probability 3/4. What is the expected number of games in a match?

Answer: $\boxed{\frac{16}{3}}$ Let the answer be E. If Tim wins the first game and Allen wins the second game or vice versa, which occurs with probability $(3/4)^2 + (1/4)^2 = 5/8$, the expected number of additional games is just E, so the expected total number of games is $E+2$. If, on the other hand, one of Tim and Allen wins both of the first two games, with probability $1-(5/8) = 3/8$, there are exactly 2 games in the match. It follows that

$$E = \frac{3}{8} \cdot 2 + \frac{5}{8} \cdot (E+2),$$

and solving gives $E = \frac{16}{3}$.

16. [**8**] The walls of a room are in the shape of a triangle ABC with $\angle ABC = 90°$, $\angle BAC = 60°$, and $AB = 6$. Chong stands at the midpoint of BC and rolls a ball toward AB. Suppose that the ball bounces off AB, then AC, then returns exactly to Chong. Find the length of the path of the ball.

Answer: $\boxed{3\sqrt{21}}$ Let C' be the reflection of C across AB and B' be the reflection of B across AC'; note that B', A, C are collinear by angle chasing. The image of the path under these reflections is just the line segment MM', where M is the midpoint of BC and M' is the midpoint of $B'C'$, so our answer is just the length of MM'. Applying the Law of Cosines to triangle $M'C'M$, we have $MM'^2 = 27 + 243 - 2 \cdot 3\sqrt{3} \cdot 9\sqrt{3} \cdot \frac{1}{2} = 189$, so $MM' = 3\sqrt{21}$.

17. [**11**] The lines $y = x$, $y = 2x$, and $y = 3x$ are the three medians of a triangle with perimeter 1. Find the length of the longest side of the triangle.

Answer: $\boxed{\frac{\sqrt{58}}{2+\sqrt{34}+\sqrt{58}}}$ The three medians of a triangle contain its vertices, so the three vertices of the triangle are (a, a), $(b, 2b)$ and $(c, 3c)$ for some a, b, and c. Then, the midpoint of (a, a) and $(b, 2b)$, which is $(\frac{a+b}{2}, \frac{a+2b}{2})$, must lie along the line $y = 3x$. Therefore,

$$\begin{aligned}\frac{a+2b}{2} &= 3 \cdot \frac{a+b}{2},\\ a + 2b &= 3a + 3b,\\ -2a &= b.\end{aligned}$$

Similarly, the midpoint of $(b, 2b)$ and $(c, 3c)$, which is $(\frac{b+c}{2}, \frac{2b+3c}{2})$, must lie along the line $y = x$. Therefore,

$$\begin{aligned}\frac{2b+3c}{2} &= \frac{b+c}{2},\\ 2b + 3c &= b + c,\\ b &= -2c,\\ c &= -\frac{1}{2}b = a.\end{aligned}$$

From this, three points can be represented as (a, a), $(-2a, -4a)$, and $(a, 3a)$. Using the distance formula, the three side lengths of the triangle are $2|a|$, $\sqrt{34}|a|$, and $\sqrt{58}|a|$. Since the perimeter of the triangle is 1, we find that $|a| = \frac{1}{2+\sqrt{34}+\sqrt{58}}$ and therefore the longest side length is $\frac{\sqrt{58}}{2+\sqrt{34}+\sqrt{58}}$.

18. [**11**] Define the sequence of positive integers $\{a_n\}$ as follows. Let $a_1 = 1, a_2 = 3$, and for each $n > 2$, let a_n be the result of expressing a_{n-1} in base $n - 1$, then reading the resulting numeral in base n, then adding 2 (in base n). For example, $a_2 = 3_{10} = 11_2$, so $a_3 = 11_3 + 2_3 = 6_{10}$. Express a_{2013} in base ten.

Answer: $\boxed{23097}$ We claim that for nonnegative integers m and for $0 \le n < 3 \cdot 2^m$, $a_{3\cdot 2^m+n} = (3 \cdot 2^m + n)(m + 2) + 2n$. We will prove this by induction; the base case for $a_3 = 6$ (when $m = 0$, $n = 0$) is given in the problem statement. Now, suppose that this is true for some pair m and n. We will divide this into two cases:

- Case 1: $n < 3 \cdot 2^m - 1$. Then, we want to prove that this is true for m and $n + 1$. In particular, writing $a_{3\cdot 2^m+n}$ in base $3 \cdot 2^m + n$ results in the digits $m + 2$ and $2n$. Consequently, reading it in base $3\cdot 2^m+n+1$ gives $a_{3\cdot 2^m+n+1} = 2+(3\cdot 2^m+n+1)(m+2)+2n = (2\cdot 2^m+n+1)(m+2)+2(n+1)$, as desired.
- Case 2: $n = 3 \cdot 2^m - 1$. Then, we want to prove that this is true for $m + 1$ and 0. Similarly to the previous case, we get that $a_{3\cdot 2^m+n+1} = a_{3\cdot 2^{m+1}} = 2 + (3 \cdot 2^m + n + 1)(m + 2) + 2n = 2 + (3 \cdot 2^{m+1})(m + 2) + 2(3 \cdot 2^m - 1) = (3 \cdot 2^{m+1} + 0)((m + 1) + 2) + 2(0)$, as desired.

In both cases, we have proved our claim.

19. [**11**] An isosceles trapezoid $ABCD$ with bases AB and CD has $AB = 13$, $CD = 17$, and height 3. Let E be the intersection of AC and BD. Circles Ω and ω are circumscribed about triangles ABE and CDE. Compute the sum of the radii of Ω and ω.

Answer: $\boxed{39}$ Let Ω have center O and radius R and let ω have center P and radius M. Let Q be the intersection of AB and OE. Note that OE is the perpendicular bisector of AB because the trapezoid is isosceles. Also, we see OE is the circumradius of Ω. On the other hand, we know by similarity of $\triangle AEB$ and $\triangle CED$ that $QE = \frac{13}{13+17} \cdot 3 = \frac{13}{30} \cdot 3$. And, because $BQ = 13/2$ and is perpendicular to OQ, we can apply the Pythagorean theorem to $\triangle OQB$ to see $OQ = \sqrt{R^2 - \left(\frac{13}{2}\right)^2}$. Since $OE = OQ + QE$, $R = \frac{13}{30} \cdot 3 + \sqrt{R^2 - \left(\frac{13}{2}\right)^2}$. Solving this equation for R yields $R = \frac{13}{30} \cdot 39$. Since by similarity $M = \frac{17}{13}R$, we know $R + M = \frac{30}{13}R$, so $R + M = 39$.

20. [**11**] The polynomial $f(x) = x^3 - 3x^2 - 4x + 4$ has three real roots r_1, r_2, and r_3. Let $g(x) = x^3 + ax^2 + bx + c$ be the polynomial which has roots s_1, s_2, and s_3, where $s_1 = r_1 + r_2 z + r_3 z^2$, $s_2 = r_1 z + r_2 z^2 + r_3$, $s_3 = r_1 z^2 + r_2 + r_3 z$, and $z = \frac{-1+i\sqrt{3}}{2}$. Find the real part of the sum of the coefficients of $g(x)$.

Answer: $\boxed{-26}$ Note that $z = e^{\frac{2\pi}{3}i} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, so that $z^3 = 1$ and $z^2 + z + 1 = 0$. Also, $s_2 = s_1 z$ and $s_3 = s_1 z^2$.

Then, the sum of the coefficients of $g(x)$ is $g(1) = (1-s_1)(1-s_2)(1-s_3) = (1-s_1)(1-s_1 z)(1-s_1 z^2) = 1 - (1 + z + z^2)s_1 + (z + z^2 + z^3)s_1^2 - z^3 s_1^3 = 1 - s_1^3$.

Meanwhile, $s_1^3 = (r_1 + r_2 z + r_3 z^2)^3 = r_1^3 + r_2^3 + r_3^3 + 3r_1^2 r_2 z + 3r_1^2 r_3 z^2 + 3r_2^2 r_3 z + 3r_2^2 r_1 z^2 + 3r_3^2 r_1 z + 3r_3^2 r_2 z^2 + 6r_1 r_2 r_3$.

Since the real parts of both z and z^2 are $-\frac{1}{2}$, and since all of r_1, r_2, and r_3 are real, the real part of s_1^3 is $r_1^3 + r_2^3 + r_3^3 - \frac{3}{2}(r_1^2 r_2 + \cdots + r_3^2 r_2) + 6r_1 r_2 r_3 = (r_1 + r_2 + r_3)^3 - \frac{9}{2}(r_1 + r_2 + r_3)(r_1 r_2 + r_2 r_3 + r_3 r_1) + \frac{27}{2} r_1 r_2 r_3 = 3^3 - \frac{9}{2} \cdot 3 \cdot -4 + \frac{27}{2} \cdot -4 = 27$.

Therefore, the answer is $1 - 27 = -26$.

21. [**14**] Find the number of positive integers $j \leq 3^{2013}$ such that

$$j = \sum_{k=0}^{m} \left((-1)^k \cdot 3^{a_k}\right)$$

for some strictly increasing sequence of nonnegative integers $\{a_k\}$. For example, we may write $3 = 3^1$ and $55 = 3^0 - 3^3 + 3^4$, but 4 cannot be written in this form.

Answer: $\boxed{2^{2013}}$ Clearly m must be even, or the sum would be negative. Furthermore, if $a_m \leq 2013$, the sum cannot exceed 3^{2013} since $j = 3^{a_m} + \sum_{k=0}^{m-1}\left((-1)^k \cdot 3^{a_k}\right) \leq 3^{a_m}$. Likewise, if $a_m > 2013$, then the sum necessarily exceeds 3^{2013}, which is not hard to see by applying the Triangle Inequality and summing a geometric series. Hence, the elements of $\{a_k\}$ can be any subset of $\{0, 1, \ldots, 2013\}$ with an odd number of elements. Since the number of even-sized subsets is equal to the number of odd-sized elements, there are $\frac{2^{2014}}{2} = 2^{2013}$ such subsets.

Now, it suffices to show that given such an $\{a_k\}$, the value of j can only be obtained in this way. Suppose for the the sake of contradiction that there exist two such sequences $\{a_k\}_{0 \leq k \leq m_a}$ and $\{b_k\}_{0 \leq k \leq m_b}$ which produce the same value of j for j positive or negative, where we choose $\{a_k\}, \{b_k\}$ such that $\min(m_a, m_b)$ is as small as possible. Then, we note that since $3^{a_0} + 3^{a_1} + \ldots + 3^{(a_{m_a}-1)} \leq 3^0 + 3^1 + \ldots + 3^{(a_{m_a}-1)} < 2(3^{(a_{m_a}-1)})$, we have that $\sum_{k=0}^{m_a}\left((-1)^k \cdot 3^{a_k}\right) > 3^{(a_{m_a}-1)}$. Similarly, we get that $3^{(a_{m_b}-1)} \geq \sum_{k=0}^{m_b}\left((-1)^k \cdot 3^{a_k}\right) > 3^{(m_b-1)}$; for the two to be equal, we must have $m_a = m_b$. However, this means that the sequences obtained by removing a_{m_a} and a_{m_b} from $\{a_k\}$ $\{b_k\}$ have smaller maximum value but still produce the same alternating sum, contradicting our original assumption.

22. [**14**] Sherry and Val are playing a game. Sherry has a deck containing 2011 red cards and 2012 black cards, shuffled randomly. Sherry flips these cards over one at a time, and before she flips each card over, Val guesses whether it is red or black. If Val guesses correctly, she wins 1 dollar; otherwise, she loses 1 dollar. In addition, Val must guess red exactly 2011 times. If Val plays optimally, what is her expected profit from this game?

Answer: $\boxed{\frac{1}{4023}}$ We will prove by induction on $r+b$ that the expected profit for guessing if there are r red cards, b black cards, and where g guesses must be red, is equal to $(b-r)+\frac{2(r-b)}{(r+b)}g$. It is not difficult to check that this holds in the cases $(r,b,g)=(1,0,0),(0,1,0),(1,0,1),(0,1,1)$. Then, suppose that this is true as long as the number of cards is strictly less than $r+b$; we will prove that it also holds true when there are r red and b blue cards.

Let $f(r,b,g)$ be her expected profit under these conditions. If she guesses red, her expected profit is

$$\frac{r}{r+b}(1+f(r-1,b,g-1))+\frac{b}{r+b}(-1+f(r,b-1,g-1))=(b-r)+\frac{2(r-b)}{(r+b)}g.$$

Similarly, if she guesses black, her expected profit is

$$\frac{r}{r+b}(-1+f(r-1,b,g))+\frac{b}{r+b}(1+f(r,b-1,g))=(b-r)+\frac{2(r-b)}{(r+b)}g.$$

Plugging in the our starting values gives an expected profit of $\frac{1}{4023}$.

23. [**14**] Let $ABCD$ be a parallelogram with $AB=8$, $AD=11$, and $\angle BAD=60^\circ$. Let X be on segment CD with $CX/XD=1/3$ and Y be on segment AD with $AY/YD=1/2$. Let Z be on segment AB such that AX, BY, and DZ are concurrent. Determine the area of triangle XYZ.

Answer: $\boxed{\frac{19\sqrt{3}}{2}}$ Let AX and BD meet at P. We have $DP/PB=DX/AB=3/4$. Now, applying Ceva's Theorem in triangle ABD, we see that

$$\frac{AZ}{ZB}=\frac{DP}{PB}\cdot\frac{AY}{YD}=\frac{3}{4}\cdot\frac{1}{2}=\frac{3}{8}.$$

Now,

$$\frac{[AYZ]}{[ABCD]}=\frac{[AYZ]}{2[ABD]}=\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{3}{11}=\frac{1}{22},$$

and similarly

$$\frac{[DYX]}{[ABCD]}=\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}=\frac{1}{4}.$$

Also,

$$\frac{[XCBZ]}{[ABCD]}=\frac{1}{2}\left(\frac{1}{4}+\frac{8}{11}\right)=\frac{43}{88}.$$

The area of XYZ is the rest of the fraction of the area of $ABCD$ not covered by the three above polygons, which by a straightforward calculation 19/88 the area of $ABCD$, so our answer is

$$8\cdot 11\cdot\sin 60^\circ\cdot\frac{19}{88}=\frac{19\sqrt{3}}{2}.$$

24. [**14**] Given a point p and a line segment l, let $d(p,l)$ be the distance between them. Let A, B, and C be points in the plane such that $AB=6$, $BC=8$, $AC=10$. What is the area of the region in the (x,y)-plane formed by the ordered pairs (x,y) such that there exists a point P inside triangle ABC with $d(P,AB)+x=d(P,BC)+y=d(P,AC)$?

Answer: $\boxed{\frac{288}{5}}$. Place ABC in the coordinate plane so that $A=(0,6),B=(0,0),C=(8,0)$. Consider a point $P=(a,b)$ inside triangle ABC. Clearly, $d(P,AB)=a,d(P,BC)=b$. Now, we see that the area of triangle ABC is $\frac{6\cdot 8}{2}=24$, but may also be computed by summing the areas of triangles PAB, PBC, PCA. The area of triangle PAB is $\frac{6\cdot a}{2}=3a$, and similarly the area of triangle PBC is $4b$. Thus, it follows easily that $d(P,CA)=\frac{24-3a-4b}{5}$. Now, we have

$$(x,y)=\left(\frac{24}{5}-\frac{8}{5}a-\frac{4}{b}b, \frac{24}{5}-\frac{3}{5}a-\frac{9}{5}b\right).$$

The desired region is the set of (x,y) obtained by those (a,b) subject to the constraints $a\geq 0, b\geq 0, 6a+8b\leq 48$.

Consequently, our region is the triangle whose vertices are obtained by evaluating (x,y) at the vertices (a,b) of the triangle. To see this, let $f(a,b)$ output the corresponding (x,y) according to the above. Then, we can write every point P in ABC as $P=m(0,0)+n(0,6)+p(8,0)$ for some $m+n+p=1$. Then, $f(P)=mf(0,0)+nf(0,6)+pf(8,0)=m(\frac{24}{5},\frac{24}{5})+n(-8,0)+p(0,-6)$, so $f(P)$ ranges over the triangle with those three vertices.

Therefore, we need the area of the triangle with vertices $\left(\frac{24}{5},\frac{24}{5}\right),(0,-6),(-8,0)$, which is easily computed (for example, using determinants) to be $\frac{288}{5}$.

25. [**17**] The sequence (z_n) of complex numbers satisfies the following properties:

- z_1 and z_2 are not real.
- $z_{n+2}=z_{n+1}^2 z_n$ for all integers $n\geq 1$.
- $\dfrac{z_{n+3}}{z_n^2}$ is real for all integers $n\geq 1$.
- $\left|\dfrac{z_3}{z_4}\right|=\left|\dfrac{z_4}{z_5}\right|=2$.

Find the product of all possible values of z_1.

Answer: $\boxed{65536}$ All complex numbers can be expressed as $r(\cos\theta+i\sin\theta)=re^{i\theta}$. Let z_n be $r_n e^{i\theta_n}$.

$\dfrac{z_{n+3}}{z_n^2}=\dfrac{z_{n+2}^2 z_{n+1}}{z_n^2}=\dfrac{z_{n+1}^5 z_n^2}{z_n^2}=z_{n+1}^5$ is real for all $n\geq 1$, so $\theta_n=\dfrac{\pi k_n}{5}$ for all $n\geq 2$, where k_n is an integer. $\theta_1+2\theta_2=\theta_3$, so we may write $\theta_1=\dfrac{\pi k_1}{5}$ with k_1 an integer.

$\dfrac{r_3}{r_4}=\dfrac{r_4}{r_5}\Rightarrow r_5=\dfrac{r_4^2}{r_3}=r_4^2 r_3$, so $r_3=1$. $\dfrac{r_3}{r_4}=2\Rightarrow r_4=\dfrac{1}{2}$, $r_4=r_3^2 r_2\Rightarrow r_2=\dfrac{1}{2}$, and $r_3=r_2^2 r_1\Rightarrow r_1=4$.

Therefore, the possible values of z_1 are the nonreal roots of the equation $x^{10}-4^{10}=0$, and the product of the eight possible values is $\dfrac{4^{10}}{4^2}=4^8=65536$. For these values of z_1, it is not difficult to construct a sequence which works, by choosing z_2 nonreal so that $|z_2|=\frac{1}{2}$.

26. [**17**] Triangle ABC has perimeter 1. Its three altitudes form the side lengths of a triangle. Find the set of all possible values of $\min(AB,BC,CA)$.

Answer: $\boxed{(\frac{3-\sqrt{5}}{4},\frac{1}{3}]}$ Let a,b,c denote the side lengths BC, CA, and AB, respectively. Without loss of generality, assume $a\leq b\leq c$; we are looking for the possible range of a.

First, note that the maximum possible value of a is $\frac{1}{3}$, which occurs when ABC is equilateral. It remains to find a lower bound for a.

Now rewrite $c=xa$ and $b=ya$, where we have $x\geq y\geq 1$. Note that for a non-equilateral triangle, $x>1$. The triangle inequality gives us $a+b>c$, or equivalently, $y>x-1$. If we let K be the area, the condition for the altitudes gives us $\frac{2K}{c}+\frac{2K}{b}>\frac{2K}{a}$, or equivalently, $\frac{1}{b}>\frac{1}{a}-\frac{1}{c}$, which after some manipulation yields $y<\frac{x}{x-1}$. Putting these conditions together yields $x-1<\frac{x}{x-1}$, and after rearranging and solving a quadratic, we get $x<\frac{3+\sqrt{5}}{2}$.

We now use the condition $a(1+x+y)=1$, and to find a lower bound for a, we need an upper bound for $1+x+y$. We know that $1+x+y<1+x+\frac{x}{x-1}=x-1+\frac{1}{x-1}+3$.

Now let $f(x)=x-1+\frac{1}{x-1}+3$. If $1<x<2$, then $1+x+y\leq 1+2x<5$. But for $x\geq 2$, we see that $f(x)$ attains a minimum of 5 at $x=2$ and continues to strictly increase after that point. Since $x<\frac{3+\sqrt{5}}{2}$, we have $f(x)<f\left(\frac{3+\sqrt{5}}{2}\right)=3+\sqrt{5}>5$, so this is a better upper bound than the case for which $1<x<2$. Therefore, $a>\left(\frac{1}{3+\sqrt{5}}\right)=\frac{3-\sqrt{5}}{4}$.

For any a such that $\sqrt{5}-2\geq a>\frac{3-\sqrt{5}}{4}$, we can let $b=\frac{1+\sqrt{5}}{2}a$ and $c=1-a-b$. For any other possible a, we can let $b=c=\frac{1-a}{2}$. The triangle inequality and the altitude condition can both be verified algebraically.

We now conclude that the set of all possible a is $\frac{3-\sqrt{5}}{4}<a\leq\frac{1}{3}$.

27. [**17**] Let W be the hypercube $\{(x_1,x_2,x_3,x_4)\mid 0\leq x_1,x_2,x_3,x_4\leq 1\}$. The intersection of W and a hyperplane parallel to $x_1+x_2+x_3+x_4=0$ is a non-degenerate 3-dimensional polyhedron. What is the maximum number of faces of this polyhedron?

Answer: $\boxed{8}$ The number of faces in the polyhedron is equal to the number of distinct cells (3-dimensional faces) of the hypercube whose interior the hyperplane intersects. However, it is possible to arrange the hyperplane such that it intersects all 8 cells. Namely, $x_1+x_2+x_3+x_4=\frac{3}{2}$ intersects all 8 cells because it passes through $(0,\frac{1}{2},\frac{1}{2},\frac{1}{2})$ (which is on the cell $x_1=0$), $(1,\frac{1}{6},\frac{1}{6},\frac{1}{6})$ (which is on the cell $x_1=1$), and the points of intersection with the other 6 cells can be found by permuting these quadruples.

28. [**17**] Let $z_0+z_1+z_2+\cdots$ be an infinite complex geometric series such that $z_0=1$ and $z_{2013}=\frac{1}{2013^{2013}}$. Find the sum of all possible sums of this series.

Answer: $\boxed{\frac{2013^{2014}}{2013^{2013}-1}}$ Clearly, the possible common ratios are the 2013 roots $r_1,r_2,\ldots,r_{2013}$ of the equation $r^{2013}=\frac{1}{2013^{2013}}$. We want the sum of the values of $x_n=\frac{1}{1-r_n}$, so we consider the polynomial whose roots are $x_1,x_2,\ldots,x_{2013}$. It is easy to see that $(1-\frac{1}{x_n})^{2013}=\frac{1}{2013^{2013}}$, so it follows that the x_n are the roots of the polynomial equation $\frac{1}{2013^{2013}}x^{2013}-(x-1)^{2013}=0$. The leading coefficient of this polynomial is $\frac{1}{2013^{2013}}-1$, and it follows easily from the Binomial Theorem that the next coefficient is 2013, so our answer is, by Vieta's Formulae,

$$-\frac{2013}{\frac{1}{2013^{2013}}-1}=\frac{2013^{2014}}{2013^{2013}-1}.$$

29. [**20**] Let $A_1,A_2,...,A_m$ be finite sets of size 2012 and let $B_1,B_2,...,B_m$ be finite sets of size 2013 such that $A_i\cap B_j=\emptyset$ if and only if $i=j$. Find the maximum value of m.

Answer: $\boxed{\binom{4025}{2012}}$ In general, we will show that if each of the sets A_i contain a elements and if each of the sets B_j contain b elements, then the maximum value for m is $\binom{a+b}{a}$.

Let U denote the union of all the sets A_i and B_j and let $|U|=n$. Consider the $n!$ orderings of the elements of U. Note that for any specific ordering, there is at most one value of i such that all the elements in A_i come before all the elements in B_i in this ordering; this follows since A_j shares at least one element with B_i and B_j shares at least one element with A_i for any other $j\neq i$.

On the other hand, the number of ways to permute the $(a+b)$ elements in $A_i\cup B_i$ so that all the elements in A_i come first is equal to $a!b!$. Therefore, the number of permutations of U where all the elements in A_i come before all the elements in B_i is equal to:

$$n!\cdot\frac{a!b!}{(a+b)!}=\frac{n!}{\binom{a+b}{a}}$$

Summing over all m values of i, the total number of orderings where, for some i, the elements in A_i come before B_i is equal to

$$\frac{n!m}{\binom{a+b}{a}}$$

But there are at most $u!$ such orderings, since there are $u!$ total orderings, so it follows that $m \leq \binom{a+b}{a}$. Equality is attained by taking U to be a set containing $(a+b)$ elements, letting A_i range over all a-element subsets of U, and letting $B_i = U \setminus A_i$ for each i.

30. [**20**] How many positive integers k are there such that

$$\frac{k}{2013}(a+b) = lcm(a,b)$$

has a solution in positive integers (a,b)?

Answer: $\boxed{1006}$ First, we can let $h = \gcd(a,b)$ so that $(a,b) = (hA, hB)$ where $\gcd(A,B) = 1$. Making these subtitutions yields $\frac{k}{2013}(hA+hB) = hAB$, so $k = \frac{2013AB}{A+B}$. Because A and B are relatively prime, $A+B$ shares no common factors with neither A nor B, so in order to have k be an integer, $A+B$ must divide 2013, and since A and B are positive, $A+B > 1$.

We first show that for different possible values of $A+B$, the values of k generated are distinct. In particular, we need to show that $\frac{2013AB}{A+B} \neq \frac{2013A'B'}{A'+B'}$ whenever $A+B \neq A'+B'$. Assume that such an equality exists, and cross-multiplying yields $AB(A'+B') = A'B'(A+B)$. Since AB is relatively prime to $A+B$, we must have $A+B$ divide $A'+B'$. With a similar argument, we can show that $A'+B'$ must divide $A+B$, so $A+B = A'+B'$.

Now, we need to show that for the same denominator $A+B$, the values of k generated are also distinct for some relatively prime non-ordered pair (A,B). Let $n = A+B = C+D$. Assume that $\frac{2013AB}{n} = \frac{2013CD}{n}$, or equivalently, $A(n-A) = C(n-C)$. After some rearrangement, we have $(C+A)(C-A) = n(C-A)$ This implies that either $C = A$ or $C = n-A = B$. But in either case, (C,D) is some permutation of (A,B).

Our answer can therefore be obtained by summing up the totients of the factors of 2013 (excluding 1) and dividing by 2 since (A,B) and (B,A) correspond to the same k value, so our answer is $\frac{2013-1}{2} = 1006$.

Remark: It can be proven that the sum of the totients of all the factors of any positive integer N equals N, but in this case, the sum of the totients can be computed by hand.

31. [**20**] Let $ABCD$ be a quadrilateral inscribed in a unit circle with center O. Suppose that $\angle AOB = \angle COD = 135°$, $BC = 1$. Let B' and C' be the reflections of A across BO and CO respectively. Let H_1 and H_2 be the orthocenters of $AB'C'$ and BCD, respectively. If M is the midpoint of OH_1, and O' is the reflection of O about the midpoint of MH_2, compute OO'.

Answer: $\boxed{\frac{1}{4}(8-\sqrt{6}-3\sqrt{2})}$ Put the diagram on the complex plane with O at the origin and A at 1. Let B have coordinate b and C have coordinate c. We obtain easily that B' is b^2, C' is c^2, and D is bc. Therefore, H_1 is $1+b^2+c^2$ and H_2 is $b+c+bc$ (we have used the fact that for triangles on the unit circle, their orthocenter is the sum of the vertices). Finally, we have that M is $\frac{1}{2}(1+b^2+c^2)$, so the reflection of O about the midpoint of MH_2 is $\frac{1}{2}(1+b^2+c^2+2b+2c+2bc) = \frac{1}{2}(b+c+1)^2$, so we just seek $\frac{1}{2}|b+c+1|^2$. But we know that $b = \text{cis}135°$ and $c = \text{cis}195°$, so we obtain that this value is $\frac{1}{4}(8-\sqrt{6}-3\sqrt{2})$.

32. [**20**] For an even integer positive integer n Kevin has a tape of length $4n$ with marks at $-2n, -2n+1, \ldots, 2n-1, 2n$. He then randomly picks n points in the set $-n, -n+1, -n+2, \ldots, n-1, n$, and places a stone on each of these points. We call a stone 'stuck' if it is on $2n$ or $-2n$, or either all the points to the right, or all the points to the left, all contain stones. Then, every minute, Kevin shifts the unstuck stones in the following manner:

- He picks an unstuck stone uniformly at random and then flips a fair coin.

• If the coin came up heads, he then moves that stone and every stone in the largest contiguous set containing that stone one point to the left. If the coin came up tails, he moves every stone in that set one point right instead.

• He repeats until all the stones are stuck.

Let p_k be the probability that at the end of the process there are exactly k stones in the right half. Evaluate

$$\frac{p_{n-1} - p_{n-2} + p_{n-3} - \ldots + p_3 - p_2 + p_1}{p_{n-1} + p_{n-2} + p_{n-3} + \ldots + p_3 + p_2 + p_1}$$

in terms of n.

Answer: $\boxed{\frac{1}{n-1}}$ After we have selected the positions of the initial n stones, we number their positions: $a_1 < a_2 < \ldots < a_n$. The conditions on how we move the stones imply that the expected value of $(a_i - a_j)$ after t minutes is still equal to $a_i - a_j$. In addition, if b_i is the final position of the ith stone, $E(b_{i+1} - b_i) = E(a_{i+1} - a_i)$. But this quantity is also equal to $(3n+2) \cdot p_i + 1 \cdot (1 - p_i)$.

Now, let's calculate the expected value of $a_{i+1} - ai$. This is the sum over $g = a_{i+1} - a_i$, and j, the number of spaces before a_i of $g \cdot \binom{j}{i-1}\binom{2n-j-g}{n-i+1}$, so we get

$$\frac{1}{\binom{2n+1}{n}} \sum_g g \cdot \sum_j \binom{j}{i-1}\binom{2n-j-g}{n-i-1}$$

But $\sum_j \binom{j}{i-1}\binom{2n-j-g}{n-i-1}$ is just $\binom{2n-g+1}{n-1}$. Therefore the expected value of $a_{i+1} - a_i$ is independent of i, so p_i is constant for all $i \neq 0, n$. It follows that the answer is $\frac{1}{n-1}$.

33. [**25**] Compute the value of $1^{25} + 2^{24} + 3^{23} + \ldots + 24^2 + 25^1$. If your answer is A and the correct answer is C, then your score on this problem will be $\left\lfloor 25 \min\left(\left(\frac{A}{C}\right)^2, \left(\frac{C}{A}\right)^2\right)\right\rfloor$.

Answer: $\boxed{66071772829247409}$ The sum is extremely unimodal, so we want to approximate it using its largest term. Taking logs of each term, we see that the max occurs when $(26-n)\log n$ peaks, and taking derivatives gives

$x + x \log x = 26$

From here it's easy to see that the answer is around 10, and slightly less (it's actually about 8.3, but in any case it's hard to find powers of anything except 10). Thus the largest term will be something like 10^{16}, which is already an order of magnitude within the desired answer 6.6×10^{16}.

To do better we'd really need to understand the behavior of the function x^{26-x}, but what approximately happens is that only the four or five largest terms in the sum are of any substantial size; thus it is reasonable here to pick some constant from 4 to 20 to multiply our guess 10^{16}; any guess between 4.0×10^{16} and 2.0×10^{17} is reasonable.

34. [**25**] For how many unordered sets $\{a, b, c, d\}$ of positive integers, none of which exceed 168, do there exist integers w, x, y, z such that $(-1)^w a + (-1)^x b + (-1)^y c + (-1)^z d = 168$? If your answer is A and the correct answer is C, then your score on this problem will be $\lfloor 25e^{-3\frac{|C-A|}{C}} \rfloor$.

Answer: $\boxed{761474}$ As an approximation, we assume a, b, c, d are ordered to begin with (so we have to divide by 24 later) and add to 168 with a unique choice of signs; then, it suffices to count $e + f + g + h = 168$ with each e, f, g, h in $[-168, 168]$ and then divide by 24 (we drop the condition that none of them can be zero because it shouldn't affect the answer that much).

One way to do this is generating functions. We want the coefficient of t^{168} in the generating function

$(t^{-168} + t^{-167} + \ldots + t^{167} + t^{168})^4 = (t^{169} - t^{-168})^4/(t-1)^4$

Clearing the negative powers, it suffices to find the coefficient of t^{840} in

$(t^{337} - 1)^4/(t-1)^4 = (1 - 4t^{337} + 6t^{674} - \ldots)\frac{1}{(t-1)^4}$.

To do this we expand the bottom as a power series in t:

$\frac{1}{(t-1)^4} = \sum_{n\geq 0} \binom{n+3}{3} t^n$

It remains to calculate $\binom{840+3}{3} - 4 \cdot \binom{840-337+3}{3} + 6 \cdot \binom{840-674+3}{3}$. This is almost exactly equal to $\frac{1}{6}(843^3 - 4 \cdot 506^3 + 6 \cdot 169^3) \approx 1.83 \times 10^7$.

Dividing by 24, we arrive at an estimation 762500. Even if we use a bad approximation $\frac{1}{6\cdot 24}(850^3 - 4 \cdot 500^3 + 6 \cdot 150^3)$ we get approximately 933000, which is fairly close to the answer.

35. [**25**] Let P be the number to partition 2013 into an ordered tuple of prime numbers? What is $\log_2(P)$? If your answer is A and the correct answer is C, then your score on this problem will be $\left\lfloor \frac{125}{2}\left(\min\left(\frac{C}{A}, \frac{A}{C}\right) - \frac{3}{5}\right)\right\rfloor$ or zero, whichever is larger.

Answer: 614.519... We use the following facts and heuristics.

(1) The ordered partitions of n into any positive integers (not just primes) is 2^{n-1}. This can be guessed by checking small cases and finding a pattern, and is not difficult to prove.

(2) The partitions of $\frac{2013}{n}$ into any positive integers equals the partitions of 2013 into integers from the set $\{n, 2n, 3n, \cdots\}$.

(3) The small numbers matter more when considering partitions.

(4) The set of primes $\{2, 3, 5, 7, \cdots\}$ is close in size (near the small numbers) to $\{3, 6, 9, \cdots\}$ or $\{2, 4, 6, \cdots\}$.

(5) The prime numbers get very sparse compared to the above two sets in the larger numbers.

Thus, using these heuristics, the number of partitions of 2013 into primes is approximately $2^{\frac{2013}{3}-1}$ or $2^{\frac{2013}{2}-1}$, which, taking logarithms, give 670 and 1005.5, respectively. By (5), we should estimate something that is slightly less than these numbers.

36. [**24**] (Mathematicians A to Z) Below are the names of 26 mathematicians, one for each letter of the alphabet. Your answer to this question should be a subset of $\{A, B, \cdots, Z\}$, where each letter represents the corresponding mathematician. If two mathematicians in your subset have birthdates that are within 20 years of each other, then your score is 0. Otherwise, your score is $\max(3(k-3), 0)$ where k is the number of elements in your subset.

Niels Abel	Isaac Netwon
Étienne Bézout	Nicole Oresme
Augustin-Louis Cauchy	Blaise Pascal
René Descartes	Daniel Quillen
Leonhard Euler	Bernhard Riemann
Pierre Fatou	Jean-Pierre Serre
Alexander Grothendieck	Alan Turing
David Hilbert	Stanislaw Ulam
Kenkichi Iwasawa	John Venn
Carl Jacobi	Andrew Wiles
Andrey Kolmogorov	Leonardo Ximenes
Joseph-Louis Lagrange	Shing-Tung Yau
John Milnor	Ernst Zermelo

Answer: {O, D, P, E, B, C, R, H, K, S, Y} A knowledgeable math historian might come up with this 11-element subset, earning 24 points: {O, D, P, E, B, C, R, H, K, S, Y}.