THE HARVARD-MIT MATHEMATICS TOURNAMENT

TEAM ROUND

This test consists of ten problems to be solved by a team in one hour. The problems are unequally weighted with point values as shown in brackets. They are *not* necessarily in order of difficulty, though harder problems are generally worth more points. We do not expect most teams to get through all the problems.

No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted. If there is a chalkboard in the room, you may write on it *during* the round only, *not* before it starts.

Numerical answers should, where applicable, be simplified as much as reasonably possible and must be exact unless otherwise specified. Correct mathematical notation must be used. Your proctor *cannot* assist you in interpreting or solving problems but has our phone number and may help with any administrative difficulties.

All problems require full written proof/justification. Even if the answer is numerical, you must prove your result.

If you believe the test contains an error, please submit your protest in writing to Lobby 10 during lunchtime.

Enjoy!

HMMT 2013
Saturday 16 February 2013

Team Round

Organization ____

Team ______________________________________ Team ID# ________

1. [**10**]

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2. [**15**]

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4. [**20**]

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5. [**25**]

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6. [**25**]

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7. [**30**]

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8. [**35**]

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9. [**35**]

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2. [**15**] A cafe has 3 tables and 5 individual counter seats. People enter in groups of size between 1 and 4, inclusive, and groups never share a table. A group of more than 1 will always try to sit at a table, but will sit in counter seats if no tables are available. Conversely, a group of 1 will always try to sit at the counter first. One morning, M groups consisting of a total of N people enter and sit down. Then, a single person walks in, and realizes that all the tables and counter seats are occupied by some person or group. What is the minimum possible value of $M+N$?

3. [**15**] Let ABC be a triangle with circumcenter O such that $AC = 7$. Suppose that the circumcircle of AOC is tangent to BC at C and intersects the line AB at A and F. Let FO intersect BC at E. Compute BE.

4. [**20**] Let a_1, a_2, a_3, a_4, a_5 be real numbers whose sum is 20. Determine with proof the smallest possible value of
$$\sum_{1\le i<j\le 5} \lfloor a_i + a_j \rfloor.$$

5. [**25**] Thaddeus is given a 2013×2013 array of integers each between 1 and 2013, inclusive. He is allowed two operations:

   1. Choose a row, and subtract 1 from each entry.
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9. [**35**] Let m be an odd positive integer greater than 1. Let S_m be the set of all non-negative integers less than m which are of the form $x+y$, where $xy-1$ is divisible by m. Let $f(m)$ be the number of elements of S_m.

   (a) Prove that $f(mn) = f(m)f(n)$ if m, n are relatively prime odd integers greater than 1.

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   (a) Prove that $f(mn) = f(m)f(n)$ if m, n are relatively prime odd integers greater than 1.

   (b) Find a closed form for $f(p^k)$, where $k > 0$ is an integer and p is an odd prime.

10. [**40**] Chim Tu has a large rectangular table. On it, there are finitely many pieces of paper with non-overlapping interiors, each one in the shape of a convex polygon. At each step, Chim Tu is allowed to slide one piece of paper in a straight line such that its interior does not touch any other piece of paper during the slide. Can Chim Tu always slide all the pieces of paper off the table in finitely many steps?