HMMT 2013

Saturday 16 February 2013

Team Round

1. [**10**] Let a and b be real numbers such that $\frac{ab}{a^2+b^2} = \frac{1}{4}$. Find all possible values of $\frac{|a^2-b^2|}{a^2+b^2}$.

   Answer: $\boxed{\frac{\sqrt{3}}{2}}$ The hypothesis statement is equivalent to

$$a^2 + b^2 = 4ab$$
$$1 : (a+b)^2 = 6ab$$
$$2 : (a-b)^2 = 2ab$$

Multiplying equations 1 and 2,

$$(a^2-b^2)^2 = 12(ab)^2$$
$$|a^2-b^2| = \pm\sqrt{12}ab$$

Since the left hand side and ab are both positive,

$$|a^2-b^2| = \sqrt{12}ab$$
$$\frac{|a^2-b^2|}{a^2+b^2} = \frac{\sqrt{12}ab}{a^2+b^2} = \frac{\sqrt{12}}{4} = \frac{\sqrt{3}}{2}$$

(It is clear that such a and b exist: for example, we can take $a = 1$ and solve for b by way of the quadratic formula.)

2. [**15**] A cafe has 3 tables and 5 individual counter seats. People enter in groups of size between 1 and 4, inclusive, and groups never share a table. A group of more than 1 will always try to sit at a table, but will sit in counter seats if no tables are available. Conversely, a group of 1 will always try to sit at the counter first. One morning, M groups consisting of a total of N people enter and sit down. Then, a single person walks in, and realizes that all the tables and counter seats are occupied by some person or group. What is the minimum possible value of $M + N$?

   Answer: $\boxed{16}$ We first show that $M + N \geq 16$. Consider the point right before the last table is occupied. We have two cases: first suppose there exists at least one open counter seat. Then, every table must contribute at least 3 to the value of $M + N$, because no groups of 1 will have taken a table with one of the counter seats open. By the end, the counter must contribute at least $5 + 2 = 7$ to $M + N$, as there must be at least two groups sitting at the counter. It follows that $M + N \geq 16$. For the second case, assume the counter is full right before the last table is taken. Then, everybody sitting at the counter must have entered as a singleton, since they entered when a table was still available. Consequently, the counter must contribute 10 to $M + N$, and each table contributes at least 2, so once again $M + N \geq 16$.

   Now, $M + N = 16$ is achievable with eight groups of one, who first fill the counter seats, then the three tables. Thus, our answer is 16.

3. [**15**] Let ABC be a triangle with circumcenter O such that $AC = 7$. Suppose that the circumcircle of AOC is tangent to BC at C and intersects the line AB at A and F. Let FO intersect BC at E. Compute BE.

   Answer: $\boxed{EB = \frac{7}{2}}$ O is the circumcenter of $\triangle ABC \implies AO = CO \implies \angle OCA = \angle OAC$. Because AC is an inscribed arc of circumcircle $\triangle AOC$, $\angle OCA = \angle OFA$. Furthermore BC is tangent to circumcircle $\triangle AOC$, so $\angle OAC = \angle OCB$. However, again using the fact that O is the circumcenter of $\triangle ABC$, $\angle OCB = \angle OBC$.

   We now have that CO bisects $\angle ACB$, so it follows that triangle $CA = CB$. Also, by AA similarity we have $EOB \sim EBF$. Thus, $EB^2 = EO \cdot EF = EC^2$ by the similarity and power of a point, so $EB = BC/2 = AC/2 = 7/2$.

4. **[20]** Let a_1, a_2, a_3, a_4, a_5 be real numbers whose sum is 20. Determine with proof the smallest possible value of

$$\sum_{1 \le i < j \le 5} \lfloor a_i + a_j \rfloor.$$

Answer: $\boxed{72}$ We claim that the minimum is 72. This can be achieved by taking $a_1 = a_2 = a_3 = a_4 = 0.4$ and $a_5 = 18.4$. To prove that this is optimal, note that

$$\sum_{1 \le i < j \le 5} \lfloor a_i + a_j \rfloor = \sum_{1 \le i < j \le 5} (a_i + a_j) - \{a_i + a_j\} = 80 - \sum_{1 \le i < j \le 5} \{a_i + a_j\},$$

so it suffices to maximize

$$\sum_{1 \le i < j \le 5} \{a_i + a_j\} = \sum_{i=1}^{5} \{a_i + a_{i+2}\} + \sum_{i=1}^{5} \{a_i + a_{i+1}\},$$

where $a_6 = a_1$ and $a_7 = a_2$, Taking each sum modulo 1, it is clear that both are integers. Thus, the above sum is at most $2 \cdot 4 = 8$, and our original expression is at least $80 - 8 = 72$, completing the proof.

5. **[25]** Thaddeus is given a 2013×2013 array of integers each between 1 and 2013, inclusive. He is allowed two operations:

1. Choose a row, and subtract 1 from each entry.
2. Chooses a column, and add 1 to each entry.

He would like to get an array where all integers are divisible by 2013. On how many arrays is this possible?

Answer: $\boxed{2013^{4025}}$ We claim that the set of grids on which it is possible to obtain an array of all zeroes $\pmod{2013}$ is indexed by ordered 4025-tuples of residues $\pmod{2013}$, corresponding to the starting entries in the first row and first column of the grid, giving the answer of 2013^{4025}. To do this, we show that given after fixing all of the entries in the first row and column, there is a unique starting grid which can become an array of all zeroes after applying the appropriate operations.

Let $a_{i,j}$ be the entry in the i-th row and the j-th column. Suppose there is a sequence of operations giving all zeroes in the array; let r_i be the number of times we operate on row i, and let c_j be the number of times we operate on column j. It is enough to take all of these values to be residues modulo 2013. Clearly, $a_{i,j} + r_i + c_j \equiv 0 \pmod{2013}$ for each i, j. In particular, $r_1 + c_1 \equiv a_{1,1}$. Now, for each i, j, we have

$$\begin{aligned} a_{i,j} &\equiv -r_i - c_j \\ &\equiv (a_{i,1} + c_1) + (a_{1,j} + r_1) \\ &\equiv a_{i,1} + a_{1,j} - a_{1,1}, \end{aligned}$$

which is fixed. Thus, there rest of the entries in the grid are forced.

Conversely, if we set $a_{i,j}$ to be the appropriate representative of the residue class of $a_{i,1} + a_{1,j} - a_{1,1}$ modulo 2013, we may take $r_i \equiv -a_{i,1} \pmod{2013}$, and $c_j \equiv a_{1,1} - a_{1,j} \pmod{2013}$ for each i, j. It is clear that $a_{i,j} + r_i + c_j \equiv 0 \pmod{2013}$ for each i, j, so we're done.

6. **[25]** Let triangle ABC satisfy $2BC = AB + AC$ and have incenter I and circumcircle ω. Let D be the intersection of AI and ω (with A, D distinct). Prove that I is the midpoint of AD.

Answer: $\boxed{\text{N/A}}$ Since AD is an angle bisector, D is the midpoint of arc BC opposite A on ω. It is well-known that B, I, and C lie on a circle centered at D. Thus $BD = DC = DI$. Applying Ptolemy's theorem to cyclic quadrilateral $ABDC$, we get

$AB \cdot DC + AC \cdot BD = AD \cdot BC = AD \cdot (AB + AC)/2$

Using $BD = DC$ we have immediately that $AD = 2BD = 2DI$ so I is the midpoint of AD as desired.

Solution 2: Let P and Q be the midpoints of AB and AC, and take the point E on segment BC such that $BE = BP$. Note that $CE = AB - BE = AB - BP = \frac{AB+AC}{2} - \frac{AB}{2} = \frac{AC}{2} = CQ$, so triangles BPE and CQE are isosceles. In addition, $\frac{BE}{EC} = \frac{AB/2}{AC/2} = \frac{AB}{AC}$, so by the angle bisector theorem, AE bisects $\angle CAB$, whence E must lie on the bisector of $\angle A$.

Since triangles BPE and CQE are isosceles, the bisectors of angles B and C are the perpendicular bisectors of segments PE and EQ, respectively. Thus, the circumcenter of $\triangle PQE$ is I, so the perpendicular bisector of PQ meets the bisector of $\angle A$ at I.

Furthermore, since $\angle DAB = \angle CAD$, arcs $\widehat{BD}$ and $\widehat{DC}$ have the same measure, so $BD = DC$, whence the perpendicular bisector of BC meets the bisector of $\angle A$ at D. A homothety centered at A with factor $1/2$ maps BC to PQ, and so maps D to I. Thus, D is the midpoint of AI.

Solution 3: Let $a = BC$, $b = CA$, and $c = AB$, let r and R denote the lengths of the inradius and circumradius of $\triangle ABC$, respectively, let E be the intersection of segments AD and BC, and let O be the circumcenter of $\triangle ABC$. I is the midpoint of chord AD if and only if $OI \perp AD$, which is true if and only if $OA^2 = OI^2 + AI^2$. By Euler's distance formula, $OI^2 = R(R - 2r)$, and by Stewart's theorem and the angle bisector theorem we can find $AI^2 = \left(\frac{b+c}{a+b+c}\right)^2 bc(1 - \left(\frac{a}{b+c}\right)^2) = \frac{bc}{3}$. Thus, it remains to show that $6Rr = bc$.

Now, we use the combine the well-known formulas $\frac{abc}{4K} = R$ and $K = rs$ to get $abc = 4Rrs$, where K is the area of $\triangle ABC$ and s is its semiperimeter. We have $bc = Rr\frac{4s}{a} = Rr\frac{2a+2(b+c)}{a} = Rr\frac{2a+2(2a)}{a} = 6Rr$, as desired.

7. [**30**] There are n children and n toys such that each child has a strict preference ordering on the toys. We want to distribute the toys: say a distribution A dominates a distribution $B \neq A$ if in A, each child receives at least as preferable of an toy as in B. Prove that if some distribution is not dominated by any other, then at least one child gets his/her favorite toy in that distribution.

   Answer: N/A Suppose we have a distribution A assigning each child C_i, $i = 1, 2, \ldots, n$, toy T_i, such that no child C_i gets their top preference $T_i' \neq T_i$. Then, pick an arbitrary child C_1 and construct the sequence of children $C_{i_1}, C_{i_2}, C_{i_3}, \ldots$ where $i_1 = 1$ and $C_{i_{k+1}}$ was assigned the favorite toy T_{i_k}' of the last child C_{i_k}. Eventually, some $C_{i_k} = C_{i_1}$; at this point, just pass the toys around this cycle so that each of these children gets their favorite toy. Clearly the resulting distribution dominates the original, so we're done.

8. [**35**] Let points A and B be on circle ω centered at O. Suppose that ω_A and ω_B are circles not containing O which are internally tangent to ω at A and B, respectively. Let ω_A and ω_B intersect at C and D such that D is inside triangle ABC. Suppose that line BC meets ω again at E and let line EA intersect ω_A at F. If $FC \perp CD$, prove that O, C, and D are collinear.

   Answer: N/A Let $H = CA \cap \omega$, and $G = BH \cap \omega_B$. There are homotheties centered at A and B taking $\omega_A \to \omega$ and $\omega_B \to \omega$ that take $A : F \mapsto E$, $A : C \mapsto H$, $B : C \mapsto E$ and $B : G \mapsto H$. In particular $CF||EH||CG$, so C, F, G are collinear, lying on a line perpendicular to DC.

   Because of the right angles at C, DF and DG are diameters of ω_A, ω_B, respectively. Also, we have that the ratio of the sizes of ω_A and ω_B, under the two homotheties above, is $CF/EH \cdot EH/CG = CF/CG$. Therefore, $DF/DG = CF/CG$, but then $\triangle DCF$ and $\triangle DCG$ are both right triangles which share one side and have hypotenuse and other side in proportion; it is obvious now that the two circles ω_A and ω_B are congruent.

   Therefore, O has the same distance to A and B, and so the same distances to the centers of the two circles as well. As a result, O lies on the radical axis CD as desired.

9. [**35**] Let m be an odd positive integer greater than 1. Let S_m be the set of all non-negative integers less than m which are of the form $x + y$, where $xy - 1$ is divisible by m. Let $f(m)$ be the number of elements of S_m.

(a) Prove that $f(mn) = f(m)f(n)$ if m, n are relatively prime odd integers greater than 1.

(b) Find a closed form for $f(p^k)$, where $k > 0$ is an integer and p is an odd prime.

Answer: N/A For a positive integer n, let $\mathbb{Z}/n\mathbb{Z}$ denote the set of residues modulo n and $(\mathbb{Z}/n\mathbb{Z})^*$ denote the set of residues modulo n that are relatively prime to n. Then, rephrased, S_m is the set of residues modulo m of the form $x + x^{-1}$, where $x \in (\mathbb{Z}/m\mathbb{Z})^*$.

For part (a), suppose $a = x + x^{-1} \in S_m$ and $b = y + y^{-1} \in S_n$. By the Chinese Remainder Theorem, there exists a residue $z \in (\mathbb{Z}/mn\mathbb{Z})^*$ such that $m|(x - z)$ and $n|(y - z)$, and thus $z + z^{-1} \equiv x + x^{-1} \pmod{m}$ and $z + z^{-1} \equiv y + y^{-1} \pmod{n}$. Therefore, all $f(m)f(n)$ residues modulo mn which result from applying the Chinese Remainder Theorem to an element each of S_m and S_n are in S_{mn}. Conversely, given $z + z^{-1} \in (\mathbb{Z}/mn\mathbb{Z})^*$, taking $z + z^{-1}$ modulo m and n gives elements of S_m and S_n, so indeed $f(mn) = f(m)f(n)$.

We now proceed to part (b). For each $x \in (\mathbb{Z}/p^k\mathbb{Z})^*$, denote $q(x)$ to be the largest non-negative integer $i \leq k$ such that p^i divides $x^2 - 1$ (this is clearly well-defined). For a given x, let $g(x)$ be the number of $y \in (\mathbb{Z}/p^k\mathbb{Z})^*$ such that $x + x^{-1} \equiv y + y^{-1} \pmod{p^k}$. Note that this condition is equivalent to $(x - y)(xy - 1) \equiv 0 \pmod{p^k}$.

First, consider the case in which $q(x) \geq k/2$, in which case we have $p^{\lceil k/2 \rceil}|(x - 1)(x + 1)$, and because p is odd, $x \equiv \pm 1 \pmod{p^{\lceil k/2 \rceil}}$. Thus, either $(x - 1)^2 \equiv 0 \pmod{p^k}$ or $(x + 1)^2 \equiv 0 \pmod{p^k}$, and it follows that $x + x^{-1} \equiv \pm 2 \pmod{p^k}$ (clearly 2 and -2 are distinct). Conversely, $x + x^{-1} \equiv \pm 2$ implies $(x \pm 1) \equiv 0 \pmod{p^{\lceil k/2 \rceil}}$, which in turn implies $q(x) \geq k/2$. It is now clear that there are exactly two elements of S_m corresponding to residues of the form $x + x^{-1}$ with $q(x) \geq k/2$, and all other elements of S_m come from x with $q(x) < k/2$.

Fix x with $q(x) < k/2$; we will compute $g(x)$. Suppose $(x - y)(xy - 1) \equiv 0 \pmod{p^k}$, and say $x - y, xy - 1$ have j, j' factors of p, respectively. If $j \leq j'$, note that $x^2 - 1 \equiv xy - 1 \equiv 0 \pmod{p^j}$, so $j \leq q(x)$. Similarly, if $j' \leq j$, $x \equiv y \pmod{p^{j'}}$, so $x^2 - 1 \equiv xy - 1 \equiv 0 \pmod{p^{j'}}$, and so $j' \leq q(x)$. It follows that $\min(j, j') \leq q(x)$, and thus $\max(j, j') \geq k - q(x)$.

Suppose $p^{k-q(x)}|(x - y)$. Then, $xy - 1 \equiv x^2 - 1 \equiv 0 \pmod{p^{q(x)}}$ because $q(x) < k/2$, so any y with $p^{k-q(x)}|(x - y)$ satisfies $(x - y)(xy - 1) \equiv 0 \pmod{p^k}$. Now, suppose $p^{k-q(x)}|(xy - 1)$, that is, $y \equiv x^{-1} \pmod{p^{k-q(x)}}$. Then, $xy \equiv 1 \pmod{p^{q(x)}}$, and since $x^2 \equiv 1 \pmod{p^{q(x)}}$, we have $x \equiv y \pmod{p^{q(x)}}$, so again we have $(x - y)(xy - 1) \equiv 0 \pmod{p^k}$. It follows that the set of y satisfying $(x - y)(xy - 1) \equiv 0 \pmod{p^k}$ is exactly the set of y with $y \equiv x \pmod{p^{k-q(x)}}$ or $y \equiv x^{-1} \pmod{p^{k-q(x)}}$. x, x^{-1} are distinct residues modulo $p^{k-q(x)}$, because $x^2 - 1$ has fewer than $k/2$ factors of p, so it follows that $g(x) = 2p^{q(x)}$. In particular, note that these values are distinct for different values of $q(x) < k/2$, so $x + x^{-1} \equiv y + y^{-1}$ implies $q(x) = q(y)$ or $q(x), q(y) \geq k/2$.

For each integer i with $0 \leq i < k/2$, we need to compute the number of $x \in (\mathbb{Z}/p^k\mathbb{Z})$ with $q(x) = i$. Clearly, this is the number of x with $p^i|x^2 - 1$ minus the number of x with $p^{i+1}|x^2 - 1$. When $i = 0$, the number of x with $p^i|x^2 - 1$ is clearly $p^{k-1}(p - 1)$, and when $i > 0$, this number is $2p^{k-i}$, as we have $x \equiv \pm 1 \pmod{p^i}$.

We can now count the number of elements of S_{p^k} using casework on the value of $q(x)$ where we take x to be such that $x + x^{-1}$ is a particular element of S_m. Applying the results from the previous two paragraphs, our answer is

$$2 + \frac{p^{k-1}(p-3)}{2} + \sum_{i=1}^{\lceil k/2 \rceil - 1} \frac{2p^{k-i} - 2p^{k-i-1}}{2p^i},$$

where the summand 2 comes from $\pm 2 \in S_{p^k}$, corresponding to all x with $q(x) > k/2$, the next summand comes from those x with $q(x) = 0$, and each additional summand comes from those x with $q(x) = i$ in the relevant range. We may evaluate the last sum as a geometric series, to obtain the final closed form answer of

$$2 + \frac{p^{k-1}(p-3)}{2} + \frac{p^{k-1} - p^{\frac{1+(-1)^k}{2}}}{p+1}.$$

10. [**40**] Chim Tu has a large rectangular table. On it, there are finitely many pieces of paper with non-overlapping interiors, each one in the shape of a convex polygon. At each step, Chim Tu is allowed to slide one piece of paper in a straight line such that its interior does not touch any other piece of paper during the slide. Can Chim Tu always slide all the pieces of paper off the table in finitely many steps?

Answer: $\boxed{\text{N/A}}$ Let the pieces of paper be $P_1, P_2, \ldots, P_n$ in the Cartesian plane. It suffices to show that for any constant distance D, they can be slid so that each pairwise distance is at least D. Then, we can apply this using D equal to the diameter of the rectangle, sliding all but at most one of the pieces of paper off the table, and then slide this last one off arbitrarily.

We show in particular that there is always a polygon P_i which can be slid arbitrarily far to the right (i.e. in the positive x-direction). For each P_i let B_i be a bottommost (i.e. lowest y-coordinate) point on the boundary of P_i. Define a point Q to be *exposed* if the ray starting at Q in the positive x direction meets the interior of no piece of paper.

Consider the set of all exposed B_i; this set is nonempty because certainly the bottommost of all the B_i is exposed. Of this set, let B_k be the exposed B_i with maximal y-coordinate, and if there are more than one such, choose the one with maximal x-coordinate. We claim that the corresponding P_k can be slid arbitrarily far to the right.

Suppose for the sake of contradiction that there is some polygon blocking this path. To be precise, if A_k is the highest point of P_k, then the region R formed by the right-side boundary of P_k and the rays pointing in the positive x direction from A_k and B_k, must contain interior point(s) of some set of polygon(s) P_j in its interior. All of their bottommost points B_j must lie in R, since none of them can have boundary intersecting the ray from B_k, by the construction of B_k.

Because B_k was chosen to be rightmost out of all the exposed B_i with that y-coordinate, it must be that all of the B_j corresponding to the blocking P_j have larger y-coordinate. Now, choose of these the one with smallest y-coordinate - it must be exposed, and it has strictly higher y-coordinate than B_k, contradiction. It follows that the interior of R intersects no pieces of paper.

Now, for a fixed D such that D is at least the largest distance between any two points of two polygons, we can shift this exposed piece of paper nD to the right, the next one of the remaining pieces $(n-1)D$, and so on, so that the pairwise distances between pieces of paper, even when projected onto the x-axis, are at least D each. We're done.