HMMT November 2013

Saturday 9 November 2013

General Test

1. [**2**] What is the smallest non-square positive integer that is the product of four prime numbers (not necessarily distinct)?

2. [**3**] Plot points A, B, C at coordinates $(0,0)$, $(0,1)$, and $(1,1)$ in the plane, respectively. Let S denote the union of the two line segments AB and BC. Let X_1 be the area swept out when Bobby rotates S *counterclockwise* 45 degrees about point A. Let X_2 be the area swept out when Calvin rotates S *clockwise* 45 degrees about point A. Find $\frac{X_1+X_2}{2}$.

3. [**4**] A 24-hour digital clock shows times $h:m:s$, where h, m, and s are integers with $0 \leq h \leq 23$, $0 \leq m \leq 59$, and $0 \leq s \leq 59$. How many times $h:m:s$ satisfy $h+m=s$?

4. [**4**] A 50-card deck consists of 4 cards labeled "i" for $i = 1, 2, \ldots, 12$ and 2 cards labeled "13". If Bob randomly chooses 2 cards from the deck without replacement, what is the probability that his 2 cards have the same label?

5. [**5**] Let ABC be an isosceles triangle with $AB = AC$. Let D and E be the midpoints of segments AB and AC, respectively. Suppose that there exists a point F on ray $\overrightarrow{DE}$ outside of ABC such that triangle BFA is similar to triangle ABC. Compute $\frac{AB}{BC}$.

6. [**5**] Find the number of positive integer divisors of 12! that leave a remainder of 1 when divided by 3.

7. [**6**] Find the largest real number λ such that $a^2 + b^2 + c^2 + d^2 \geq ab + \lambda bc + cd$ for all real numbers a, b, c, d.

8. [**6**] How many of the first 1000 positive integers can be written as the sum of finitely many distinct numbers from the sequence $3^0, 3^1, 3^2, \ldots$?

9. [**7**] Let ABC be a triangle and D a point on BC such that $AB = \sqrt{2}$, $AC = \sqrt{3}$, $\angle BAD = 30°$, and $\angle CAD = 45°$. Find AD.

10. [**8**] How many functions $f : \{1, 2, \ldots, 2013\} \to \{1, 2, \ldots, 2013\}$ satisfy $f(j) < f(i) + j - i$ for all integers i, j such that $1 \leq i < j \leq 2013$?