HMMT November 2013

Saturday 9 November 2013

General Test

1. [**2**] What is the smallest non-square positive integer that is the product of four prime numbers (not necessarily distinct)?

   Answer: $\boxed{24}$ The smallest two integers that are the product of four primes are $2^4 = 16$ and $2^3 \cdot 3 = 24$. Since 16 is a perfect square and 24 is not, the answer is 24.

2. [**3**] Plot points A, B, C at coordinates $(0,0)$, $(0,1)$, and $(1,1)$ in the plane, respectively. Let S denote the union of the two line segments AB and BC. Let X_1 be the area swept out when Bobby rotates S *counterclockwise* 45 degrees about point A. Let X_2 be the area swept out when Calvin rotates S *clockwise* 45 degrees about point A. Find $\frac{X_1+X_2}{2}$.

   Answer: $\boxed{\frac{\pi}{4}}$ It's easy to see $X_1 = X_2$. Simple cutting and pasting shows that X_1 equals the area of $\frac{1}{8}$ of a circle with radius $AC = \sqrt{2}$, so $\frac{X_1+X_2}{2} = X_1 = \frac{1}{8}\pi(\sqrt{2})^2 = \frac{\pi}{4}$.

3. [**4**] A 24-hour digital clock shows times $h:m:s$, where h, m, and s are integers with $0 \le h \le 23$, $0 \le m \le 59$, and $0 \le s \le 59$. How many times $h:m:s$ satisfy $h + m = s$?

   Answer: $\boxed{1164}$ We are solving $h + m = s$ in $0 \le s \le 59$, $0 \le m \le 59$, and $0 \le h \le 23$. If $s \ge 24$, each h corresponds to exactly 1 solution, so we get $24(59-23) = 24(36)$ in this case. If $s \le 23$, we want the number of nonnegative integer solutions to $h + m \le 23$, which by lattice point counting (or balls and urns) is $\binom{23+2}{2} = (23+2)(23+1)/2 = 25 \cdot 12$. Thus our total is $12(72+25) = 12(100-3) = 1164$.

4. [**4**] A 50-card deck consists of 4 cards labeled "i" for $i = 1, 2, \ldots, 12$ and 2 cards labeled "13". If Bob randomly chooses 2 cards from the deck without replacement, what is the probability that his 2 cards have the same label?

   Answer: $\boxed{\frac{73}{1225}}$ All pairs of distinct cards (where we distinguish cards even with the same label) are equally likely. There are $\binom{2}{2} + 12\binom{4}{2} = 73$ pairs of cards with the same label and $\binom{50}{2} = 100 \cdot \frac{49}{4} = 1225$ pairs of cards overall, so the desired probability is $\frac{73}{1225}$.

5. [**5**] Let ABC be an isosceles triangle with $AB = AC$. Let D and E be the midpoints of segments AB and AC, respectively. Suppose that there exists a point F on ray $\overrightarrow{DE}$ outside of ABC such that triangle BFA is similar to triangle ABC. Compute $\frac{AB}{BC}$.

   Answer: $\boxed{\sqrt{2}}$ Let $\alpha = \angle ABC = \angle ACB$, $AB = 2x$, and $BC = 2y$, so $AD = DB = AE = EC = x$ and $DE = y$. Since $\triangle BFA \sim \triangle ABC$ and $BA = AC$, we in fact have $\triangle BFA \cong \triangle ABC$, so $BF = BA = 2x$, $FA = 2y$, and $\angle DAF = \alpha$. But $DE \parallel BC$ yields $\angle ADF = \angle ABC = \alpha$ as well, whence $\triangle FAD \sim \triangle ABC$ gives $\frac{2y}{x} = \frac{FA}{AD} = \frac{AB}{BC} = \frac{2x}{2y} \implies \frac{AB}{BC} = \frac{x}{y} = \sqrt{2}$.

6. [**5**] Find the number of positive integer divisors of 12! that leave a remainder of 1 when divided by 3.

   Answer: $\boxed{66}$ First we factor $12! = 2^{10}3^5 5^2 7^1 11^1$, and note that $2, 5, 11 \equiv -1 \pmod 3$ while $7 \equiv 1 \pmod 3$. The desired divisors are precisely $2^a 5^b 7^c 11^d$ with $0 \le a \le 10$, $0 \le b \le 2$, $0 \le c \le 1$, $0 \le d \le 1$, and $a + b + d$ even. But then for any choice of a, b, exactly one $d \in \{0, 1\}$ makes $a + b + d$ even, so we have exactly one $1 \pmod 3$-divisor for every triple (a, b, c) satisfying the inequality constraints. This gives a total of $(10+1)(2+1)(1+1) = 66$.

7. [**6**] Find the largest real number λ such that $a^2 + b^2 + c^2 + d^2 \ge ab + \lambda bc + cd$ for all real numbers a, b, c, d.

   Answer: $\boxed{\frac{3}{2}}$ Let $f(a,b,c,d) = (a^2+b^2+c^2+d^2) - (ab+\lambda bc+cd)$. For fixed (b,c,d), f is minimized at $a = \frac{b}{2}$, and for fixed (a,b,c), f is minimized at $d = \frac{c}{2}$, so simply we want the largest λ such that $f(\frac{b}{2}, b, c, \frac{c}{2}) = \frac{3}{4}(b^2+c^2) - \lambda bc$ is always nonnegative. By AM-GM, this holds if and only if $\lambda \le 2\frac{3}{4} = \frac{3}{2}$.

8. [**6**] How many of the first 1000 positive integers can be written as the sum of finitely many distinct numbers from the sequence $3^0, 3^1, 3^2, \ldots$?

Answer: $\boxed{105}$ We want to find which integers have only 0's and 1's in their base 3 representation. Note that $1000_{10} = 1101001_3$. We can construct a bijection from all such numbers to the binary strings, by mapping $x_3 \leftrightarrow x_2$. Since $1101001_2 = 105_{10}$, we conclude that the answer is 105.

9. [**7**] Let ABC be a triangle and D a point on BC such that $AB = \sqrt{2}$, $AC = \sqrt{3}$, $\angle BAD = 30°$, and $\angle CAD = 45°$. Find AD.

Answer: $\boxed{\frac{\sqrt{6}}{2} \text{ OR } \frac{\sqrt{3}}{\sqrt{2}}}$ Note that $[BAD] + [CAD] = [ABC]$. If $\alpha_1 = \angle BAD$, $\alpha_2 = \angle CAD$, then we deduce $\frac{\sin(\alpha_1+\alpha_2)}{AD} = \frac{\sin\alpha_1}{AC} + \frac{\sin\alpha_2}{AB}$ upon division by $AB \cdot AC \cdot AD$. Now

$$AD = \frac{\sin(30° + 45°)}{\frac{\sin 30°}{\sqrt{3}} + \frac{\sin 45°}{\sqrt{2}}}.$$

But $\sin(30° + 45°) = \sin 30° \cos 45° + \sin 45° \cos 30° = \sin 30° \frac{1}{\sqrt{2}} + \sin 45° \frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2}(\frac{\sin 30°}{\sqrt{3}} + \frac{\sin 45°}{\sqrt{2}})$, so our answer is $\frac{\sqrt{6}}{2}$.

10. [**8**] How many functions $f : \{1, 2, \ldots, 2013\} \to \{1, 2, \ldots, 2013\}$ satisfy $f(j) < f(i) + j - i$ for all integers i, j such that $1 \le i < j \le 2013$?

Answer: $\boxed{\binom{4025}{2013}}$ Note that the given condition is equivalent to $f(j) - j < f(i) - i$ for all $1 \le i < j \le 2013$. Let $g(i) = f(i) - i$, so that the condition becomes $g(j) < g(i)$ for $i < j$ and $1 - i \le g(i) \le 2013 - i$. However, since g is decreasing, we see by induction that $g(i + 1)$ is in the desired range so long as $g(i)$ is in the desired range. Hence, it suffices to choose 2013 values for $g(1), \ldots, g(2013)$ in decreasing order from $[-2012, 2012]$, for a total of $\binom{4025}{2013}$ possible functions.