HMMT NOVEMBER 2013, 9 NOVEMBER 2013 — GUTS ROUND

1. [**5**] Evaluate $2 + 5 + 8 + \cdots + 101$.

2. [**5**] Two fair six-sided dice are rolled. What is the probability that their sum is at least 10?

3. [**5**] A square is inscribed in a circle of radius 1. Find the perimeter of the square.

HMMT NOVEMBER 2013, 9 NOVEMBER 2013 — GUTS ROUND

4. [**6**] Find the minimum possible value of $(x^2 + 6x + 2)^2$ over all real numbers x.

5. [**6**] How many positive integers less than 100 are relatively prime to 200? (Two numbers are *relatively prime* if their greatest common factor is 1.)

6. [**6**] A right triangle has area 5 and a hypotenuse of length 5. Find its perimeter.

HMMT NOVEMBER 2013, 9 NOVEMBER 2013 — GUTS ROUND

7. [**7**] Marty and three other people took a math test. Everyone got a non-negative integer score. The average score was 20. Marty was told the average score and concluded that everyone else scored below average. What was the minimum possible score Marty could have gotten in order to definitively reach this conclusion?

8. [**7**] Evaluate the expression

$$\cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cdots \cfrac{1}{2 - \cfrac{1}{2}}}}},$$

where the digit 2 appears 2013 times.

9. [**7**] Find the remainder when $1^2 + 3^2 + 5^2 + \cdots + 99^2$ is divided by 1000.

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10. [**8**] How many pairs of real numbers (x, y) satisfy the equation

$$y^4 - y^2 = xy^3 - xy = x^3y - xy = x^4 - x^2 = 0?$$

11. [**8**] David has a unit triangular array of 10 points, 4 on each side. A *looping path* is a sequence $A_1, A_2, \ldots, A_{10}$ containing each of the 10 points exactly once, such that A_i and A_{i+1} are adjacent (exactly 1 unit apart) for $i = 1, 2, \ldots, 10$. (Here $A_{11} = A_1$.) Find the number of looping paths in this array.

12. [**8**] Given that $62^2 + 122^2 = 18728$, find positive integers (n, m) such that $n^2 + m^2 = 9364$.

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13. [**9**] Let $S = \{1, 2, \ldots, 2013\}$. Find the number of ordered triples (A, B, C) of subsets of S such that $A \subseteq B$ and $A \cup B \cup C = S$.

14. [**9**] Find all triples of positive integers (x, y, z) such that $x^2 + y - z = 100$ and $x + y^2 - z = 124$.

15. [**9**] Find all real numbers x between 0 and 360 such that $\sqrt{3}\cos 10^\circ = \cos 40^\circ + \sin x^\circ$.

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16. [**10**] A bug is on one exterior vertex of solid S, a $3 \times 3 \times 3$ cube that has its center $1 \times 1 \times 1$ cube removed, and wishes to travel to the opposite exterior vertex. Let O denote the *outer* surface of S (formed by the surface of the $3 \times 3 \times 3$ cube). Let $L(S)$ denote the length of the shortest path through S. (Note that such a path cannot pass through the missing center cube, which is empty space.) Let $L(O)$ denote the length of the shortest path through O. What is the ratio $\frac{L(S)}{L(O)}$?

17. [**10**] Find the sum of $\frac{1}{n}$ over all positive integers n with the property that the decimal representation of $\frac{1}{n}$ terminates.

18. [**10**] The rightmost nonzero digit in the decimal expansion of 101! is the same as the rightmost nonzero digit of $n!$, where n is an integer greater than 101. Find the smallest possible value of n.

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19. [**11**] Let p, q, r, s be distinct primes such that $pq - rs$ is divisible by 30. Find the minimum possible value of $p + q + r + s$.

20. [**11**] There exist unique nonnegative integers A, B between 0 and 9, inclusive, such that

$$(1001 \cdot A + 110 \cdot B)^2 = 57,108,249.$$

Find $10 \cdot A + B$.

21. [**11**] Suppose A, B, C, and D are four circles of radius $r > 0$ centered about the points $(0, r)$, $(r, 0)$, $(0, -r)$, and $(-r, 0)$ in the plane. Let O be a circle centered at $(0, 0)$ with radius $2r$. In terms of r, what is the area of the union of circles A, B, C, and D subtracted by the area of circle O that is not contained in the union of A, B, C, and D?

(The *union* of two or more regions in the plane is the set of points lying in at least one of the regions.)

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22. [**12**] Let S be a subset of $\{1, 2, 3, \dots, 12\}$ such that it is impossible to partition S into k disjoint subsets, each of whose elements sum to the same value, for any integer $k \geq 2$. Find the maximum possible sum of the elements of S.

23. [**12**] The number $989 \cdot 1001 \cdot 1007 + 320$ can be written as the product of three distinct primes p, q, r with $p < q < r$. Find (p, q, r).

24. [**12**] Find the number of subsets S of $\{1, 2, \dots 6\}$ satisfying the following conditions:

- S is non-empty.
- No subset of S has the property that the sum of its elements is 10.

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25. [**13**] Let a, b be positive reals with $a > b > \frac{1}{2}a$. Place two squares of side lengths a, b next to each other, such that the larger square has lower left corner at $(0, 0)$ and the smaller square has lower left corner at $(a, 0)$. Draw the line passing through $(0, a)$ and $(a + b, 0)$. The region in the two squares lying *above* the line has area 2013. If (a, b) is the unique pair maximizing $a + b$, compute $\frac{a}{b}$.

26. [**13**] Trapezoid $ABCD$ is inscribed in the parabola $y = x^2$ such that $A = (a, a^2)$, $B = (b, b^2)$, $C = (-b, b^2)$, and $D = (-a, a^2)$ for some positive reals a, b with $a > b$. If $AD + BC = AB + CD$, and $AB = \frac{3}{4}$, what is a?

27. [**13**] Find all triples of real numbers (a, b, c) such that $a^2 + 2b^2 - 2bc = 16$ and $2ab - c^2 = 16$.

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28. [**15**] Triangle ABC has $AB = 4$, $BC = 3$, and a right angle at B. Circles ω_1 and ω_2 of equal radii are drawn such that ω_1 is tangent to AB and AC, ω_2 is tangent to BC and AC, and ω_1 is tangent to ω_2. Find the radius of ω_1.

29. [**15**] Let $\triangle XYZ$ be a right triangle with $\angle XYZ = 90°$. Suppose there exists an infinite sequence of equilateral triangles $X_0Y_0T_0, X_1Y_1T_1, \ldots$ such that $X_0 = X, Y_0 = Y$, X_i lies on the segment XZ for all $i \geq 0$, Y_i lies on the segment YZ for all $i \geq 0$, X_iY_i is perpendicular to YZ for all $i \geq 0$, T_i and Y are separated by line XZ for all $i \geq 0$, and X_i lies on segment $Y_{i-1}T_{i-1}$ for $i \geq 1$.

    Let $\mathcal{P}$ denote the union of the equilateral triangles. If the area of $\mathcal{P}$ is equal to the area of XYZ, find $\frac{XY}{YZ}$.

30. [**15**] Find the number of ordered triples of integers (a, b, c) with $1 \leq a, b, c \leq 100$ and $a^2b + b^2c + c^2a = ab^2 + bc^2 + ca^2$.

31. [**17**] Chords $\overline{AB}$ and $\overline{CD}$ of circle ω intersect at E such that $AE = 8$, $BE = 2$, $CD = 10$, and $\angle AEC = 90^\circ$. Let R be a rectangle inside ω with sides parallel to $\overline{AB}$ and $\overline{CD}$, such that no point in the interior of R lies on $\overline{AB}$, $\overline{CD}$, or the boundary of ω. What is the maximum possible area of R?

32. [**17**] Suppose that x and y are chosen randomly and uniformly from $(0, 1)$. What is the probability that $\left\lfloor \sqrt{\frac{x}{y}} \right\rfloor$ is even? *Hint:* $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

33. [**17**] On each side of a 6 by 8 rectangle, construct an equilateral triangle with that side as one edge such that the interior of the triangle intersects the interior of the rectangle. What is the total area of all regions that are contained in exactly 3 of the 4 equilateral triangles?

34. [**20**] Find the number of positive integers less than 1000000 that are divisible by some perfect cube greater than 1. Your score will be $\max\left\{0, \lfloor 20 - 200|1 - \frac{k}{S}|\rfloor\right\}$, where k is your answer and S is the actual answer.

35. [**20**] Consider the following 4 by 4 grid with one corner square removed:

    You may start at any square in this grid and at each move, you may either stop or travel to an adjacent square (sharing a side, not just a corner) that you have not already visited (the square you start at is automatically marked as visited). Determine the distinct number of paths you can take. Your score will be $\max\left\{0, \lfloor 20 - 200|1 - \frac{k}{S}|\rfloor\right\}$, where k is your answer and S is the actual answer.

36. [**20**] Pick a subset of at least four of the following seven numbers, order them from least to greatest, and write down their **labels** (corresponding letters from A through G) in that order: (A) π; (B) $\sqrt{2} + \sqrt{3}$; (C) $\sqrt{10}$; (D) $\frac{355}{113}$; (E) $16\tan^{-1}\frac{1}{5} - 4\tan^{-1}\frac{1}{240}$; (F) $\ln(23)$; and (G) $2^{\sqrt{e}}$. If the ordering of the numbers you picked is correct and you picked at least 4 numbers, then your score for this problem will be $(N - 2)(N - 3)$, where N is the size of your subset; otherwise, your score is 0.