HMMT November 2013

Saturday 9 November 2013

Team Round

Expected Value

1. [**3**] Tim the Beaver can make three different types of geometrical figures: squares, regular hexagons, and regular octagons. Tim makes a random sequence $F_0, F_1, F_2, F_3, \ldots$ of figures as follows:

   - F_0 is a square.
   - For every positive integer i, F_i is randomly chosen to be one of the 2 figures *distinct from* F_{i-1} (each chosen with equal probability $\frac{1}{2}$).
   - Tim takes 4 seconds to make squares, 6 to make hexagons, and 8 to make octagons. He makes one figure after another, with no breaks in between.

   Suppose that exactly 17 seconds after he starts making F_0, Tim is making a figure with n sides. What is the expected value of n?

2. [**4**] Gary plays the following game with a fair n-sided die whose faces are labeled with the positive integers between 1 and n, inclusive: if $n = 1$, he stops; otherwise he rolls the die, and starts over with a k-sided die, where k is the number his n-sided die lands on. (In particular, if he gets $k = 1$, he will stop rolling the die.) If he starts out with a 6-sided die, what is the expected number of rolls he makes?

3. [**6**] The digits $1, 2, 3, 4, 5, 6$ are randomly chosen (without replacement) to form the three-digit numbers $M = \overline{ABC}$ and $N = \overline{DEF}$. For example, we could have $M = 413$ and $N = 256$. Find the expected value of $M \cdot N$.

Power of a Point

4. [**4**] Consider triangle ABC with side lengths $AB = 4$, $BC = 7$, and $AC = 8$. Let M be the midpoint of segment AB, and let N be the point on the interior of segment AC that also lies on the circumcircle of triangle MBC. Compute BN.

5. [**4**] In triangle ABC, $\angle BAC = 60^\circ$. Let ω be a circle tangent to segment AB at point D and segment AC at point E. Suppose ω intersects segment BC at points F and G such that F lies in between B and G. Given that $AD = FG = 4$ and $BF = \frac{1}{2}$, find the length of CG.

6. [**6**] Points A, B, C lie on a circle ω such that BC is a diameter. AB is extended past B to point B' and AC is extended past C to point C' such that line $B'C'$ is parallel to BC and tangent to ω at point D. If $B'D = 4$ and $C'D = 6$, compute BC.

7. [**7**] In equilateral triangle ABC, a circle ω is drawn such that it is tangent to all three sides of the triangle. A line is drawn from A to point D on segment BC such that AD intersects ω at points E and F. If $EF = 4$ and $AB = 8$, determine $|AE - FD|$.

Periodicity

8. [**2**] Define the sequence $\{x_i\}_{i \geq 0}$ by $x_0 = x_1 = x_2 = 1$ and $x_k = \frac{x_{k-1} + x_{k-2} + 1}{x_{k-3}}$ for $k > 2$. Find x_{2013}.

9. [**7**] For an integer $n \geq 0$, let $f(n)$ be the smallest possible value of $|x + y|$, where x and y are integers such that $3x - 2y = n$. Evaluate $f(0) + f(1) + f(2) + \cdots + f(2013)$.

10. [**7**] Let $\omega = \cos\frac{2\pi}{727} + i\sin\frac{2\pi}{727}$. The imaginary part of the complex number

$$\prod_{k=8}^{13}\left(1 + \omega^{3^{k-1}} + \omega^{2\cdot 3^{k-1}}\right)$$

is equal to $\sin\alpha$ for some angle α between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, inclusive. Find α.