HMMT November 2013

Saturday 9 November 2013

Team Round

1. [**3**] Tim the Beaver can make three different types of geometrical figures: squares, regular hexagons, and regular octagons. Tim makes a random sequence $F_0, F_1, F_2, F_3, \ldots$ of figures as follows:

   - F_0 is a square.
   - For every positive integer i, F_i is randomly chosen to be one of the 2 figures *distinct from* F_{i-1} (each chosen with equal probability $\frac{1}{2}$).
   - Tim takes 4 seconds to make squares, 6 to make hexagons, and 8 to make octagons. He makes one figure after another, with no breaks in between.

   Suppose that exactly 17 seconds after he starts making F_0, Tim is making a figure with n sides. What is the expected value of n?

   Answer: $\boxed{7}$ We write $F_i = n$ as shorthand for "the ith figure is an n-sided polygon."

   If $F_1 = 8$, the $F_2 = 6$ or $F_2 = 4$. If $F_2 = 6$, Tim is making a 6-gon at time 13 (probability contribution 1/4). If $F_2 = 4$, $F_3 = 6$ or $F_3 = 8$ will take the time 13 mark (1/8 contribution each).

   If $F_1 = 6$, $F_2 = 8$ or $F_2 = 4$. If $F_2 = 8$, it takes the 13 mark (1/4 contribution). If $F_2 = 4$, $F_3 = 6$ or $F_3 = 8$ will take the 13 mark (1/8 contribution each).

   Thus, the expected value of the number of sides at time 13 is $0(4) + (\frac{1}{4} + \frac{1}{8} + \frac{1}{8})(6) + (\frac{1}{8} + \frac{1}{4} + \frac{1}{8})(8) = 7$.

2. [**4**] Gary plays the following game with a fair n-sided die whose faces are labeled with the positive integers between 1 and n, inclusive: if $n = 1$, he stops; otherwise he rolls the die, and starts over with a k-sided die, where k is the number his n-sided die lands on. (In particular, if he gets $k = 1$, he will stop rolling the die.) If he starts out with a 6-sided die, what is the expected number of rolls he makes?

   Answer: $\boxed{\frac{197}{60}}$ If we let a_n be the expected number of rolls starting with an n-sided die, we see immediately that $a_1 = 0$, and $a_n = 1 + \frac{1}{n}\sum_{i=1}^{n} a_i$ for $n > 1$. Thus $a_2 = 2$, and for $n \geq 3$, $a_n = 1 + \frac{1}{n}a_n + \frac{n-1}{n}(a_{n-1} - 1)$, or $a_n = a_{n-1} + \frac{1}{n-1}$. Thus $a_n = 1 + \sum_{i=1}^{n-1} \frac{1}{i}$ for $n \geq 2$, so $a_6 = 1 + \frac{60+30+20+15+12}{60} = \frac{197}{60}$.

3. [**6**] The digits $1, 2, 3, 4, 5, 6$ are randomly chosen (without replacement) to form the three-digit numbers $M = \overline{ABC}$ and $N = \overline{DEF}$. For example, we could have $M = 413$ and $N = 256$. Find the expected value of $M \cdot N$.

   Answer: $\boxed{143745}$ By linearity of expectation and symmetry,

   $$\mathbb{E}[MN] = \mathbb{E}[(100A + 10B + C)(100D + 10E + F)] = 111^2 \cdot \mathbb{E}[AD].$$

   Since

   $$\mathbb{E}[AD] = \frac{(1+2+3+4+5+6)^2 - (1^2+2^2+3^2+4^2+5^2+6^2)}{6 \cdot 5} = \frac{350}{30},$$

   our answer is $111 \cdot 35 \cdot 37 = 111 \cdot 1295 = 143745$.

4. [**4**] Consider triangle ABC with side lengths $AB = 4$, $BC = 7$, and $AC = 8$. Let M be the midpoint of segment AB, and let N be the point on the interior of segment AC that also lies on the circumcircle of triangle MBC. Compute BN.

   Answer: $\boxed{\frac{\sqrt{210}}{4} \text{ OR } \frac{\sqrt{105}}{2\sqrt{2}}}$ Let $\angle BAC = \theta$. Then, $\cos\theta = \frac{4^2+8^2-7^2}{2 \cdot 4 \cdot 8}$. Since $AM = \frac{4}{2} = 2$, and power of a point gives $AM \cdot AB = AN \cdot AC$, we have $AN = \frac{2 \cdot 4}{8} = 1$, so $NC = 8 - 1 = 7$. Law of cosines on triangle BAN gives

   $$BN^2 = 4^2 + 1^2 - 2 \cdot 4 \cdot 1 \cdot \frac{4^2+8^2-7^2}{2 \cdot 4 \cdot 8} = 17 - \frac{16+15}{8} = 15 - \frac{15}{8} = \frac{105}{8},$$

   so $BN = \frac{\sqrt{210}}{4}$.

5. [**4**] In triangle ABC, $\angle BAC = 60^\circ$. Let ω be a circle tangent to segment AB at point D and segment AC at point E. Suppose ω intersects segment BC at points F and G such that F lies in between B and G. Given that $AD = FG = 4$ and $BF = \frac{1}{2}$, find the length of CG.

Answer: $\boxed{\frac{16}{5}}$ Let $x = CG$. First, by power of a point, $BD = \sqrt{BF(BF+FG)} = \frac{3}{2}$, and $CE = \sqrt{x(x+4)}$. By the law of cosines, we have

$$(x+\frac{9}{2})^2 = \left(\frac{11}{2}\right)^2 + (4+\sqrt{x(x+4)})^2 - \frac{11}{2}(4+\sqrt{x(x+4)}),$$

which rearranges to $2(5x-4) = 5\sqrt{x(x+4)}$. Squaring and noting $x > \frac{4}{5}$ gives $(5x-16)(15x-4) = 0 \implies x = \frac{16}{5}$.

6. [**6**] Points A, B, C lie on a circle ω such that BC is a diameter. AB is extended past B to point B' and AC is extended past C to point C' such that line $B'C'$ is parallel to BC and tangent to ω at point D. If $B'D = 4$ and $C'D = 6$, compute BC.

Answer: $\boxed{\frac{24}{5}}$ Let $x = AB$ and $y = AC$, and define $t > 0$ such that $BB' = tx$ and $CC' = ty$. Then $10 = B'C' = (1+t)\sqrt{x^2+y^2}$, $4^2 = t(1+t)x^2$, and $6^2 = t(1+t)y^2$ (by power of a point), so $52 = 4^2+6^2 = t(1+t)(x^2+y^2)$ gives $\frac{13}{25} = \frac{52}{10^2} = \frac{t(1+t)}{(1+t)^2} = \frac{t}{1+t} \implies t = \frac{13}{12}$. Hence $BC = \sqrt{x^2+y^2} = \frac{10}{1+t} = \frac{10}{25/12} = \frac{24}{5}$.

7. [**7**] In equilateral triangle ABC, a circle ω is drawn such that it is tangent to all three sides of the triangle. A line is drawn from A to point D on segment BC such that AD intersects ω at points E and F. If $EF = 4$ and $AB = 8$, determine $|AE - FD|$.

Answer: $\boxed{\frac{4}{\sqrt{5}} \text{ OR } \frac{4\sqrt{5}}{5}}$ Without loss of generality, A, E, F, D lie in that order. Let $x = AE$, $y = DF$. By power of a point, $x(x+4) = 4^2 \implies x = 2\sqrt{5}-2$, and $y(y+4) = (x+4+y)^2 - (4\sqrt{3})^2 \implies y = \frac{48-(x+4)^2}{2(x+2)} = \frac{12-(1+\sqrt{5})^2}{\sqrt{5}}$. It readily follows that $x - y = \frac{4}{\sqrt{5}} = \frac{4\sqrt{5}}{5}$.

8. [**2**] Define the sequence $\{x_i\}_{i\geq 0}$ by $x_0 = x_1 = x_2 = 1$ and $x_k = \frac{x_{k-1}+x_{k-2}+1}{x_{k-3}}$ for $k > 2$. Find x_{2013}.

Answer: $\boxed{9}$ We have $x_3 = \frac{1+1+1}{1} = 3$, $x_4 = \frac{3+1+1}{1} = 5$, $x_5 = \frac{5+3+1}{1} = 9$, $x_6 = \frac{9+5+1}{3} = 5$. By the symmetry of our recurrence (or just further computation—it doesn't matter much), $x_7 = 3$ and $x_8 = x_9 = x_{10} = 1$, so our sequence has period 8. Thus $x_{2013} = x_{13} = x_5 = 9$.

9. [**7**] For an integer $n \geq 0$, let $f(n)$ be the smallest possible value of $|x+y|$, where x and y are integers such that $3x - 2y = n$. Evaluate $f(0) + f(1) + f(2) + \cdots + f(2013)$.

Answer: $\boxed{2416}$ First, we can use $3x - 2y = n$ to get $x = \frac{n+2y}{3}$. Thus $|x+y| = |\frac{n+5y}{3}|$. Given a certain n, the only restriction on y is that $3 \mid n+2y \iff 3 \mid n+5y$. Hence the set of possible $x+y$ equals the set of integers of the form $\frac{n+5y}{3}$, which in turn equals the set of integers congruent to $3^{-1}n \equiv 2n \pmod 5$. (Prove this!)

Thus $f(n) = |x+y|$ is minimized when $x+y$ equals the *least absolute remainder* $(2n)_5$ when $2n$ is divided by 5, i.e. the number between -2 and 2 (inclusive) congruent to $2n$ modulo 5. We immediately find $f(n) = f(n+5m)$ for all integers m, and the following initial values of f: $f(0) = |(0)_5| = 0$; $f(1) = |(2)_5| = 2$; $f(2) = |(4)_5| = 1$; $f(3) = |(6)_5| = 1$; and $f(4) = |(8)_5| = 2$.

Since $2013 = 403 \cdot 5 - 2$, it follows that $f(0)+f(1)+\cdots+f(2013) = 403[f(0)+f(1)+\cdots+f(4)] - f(2014) = 403 \cdot 6 - 2 = 2416$.

10. [**7**] Let $\omega = \cos\frac{2\pi}{727} + i\sin\frac{2\pi}{727}$. The imaginary part of the complex number

$$\prod_{k=8}^{13}\left(1+\omega^{3^{k-1}}+\omega^{2\cdot 3^{k-1}}\right)$$

is equal to $\sin\alpha$ for some angle α between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, inclusive. Find α.

Answer: $\boxed{\frac{12\pi}{727}}$ Note that $727 = 3^6 - 2$. Our product telescopes to $\frac{1-\omega^{3^{13}}}{1-\omega^{3^7}} = \frac{1-\omega^{12}}{1-\omega^6} = 1 + \omega^6$, which has imaginary part $\sin\frac{12\pi}{727}$, giving $\alpha = \frac{12\pi}{727}$.