HMMT November 2013

Saturday 9 November 2013

Theme Round

Traveling

1. [**2**] Two cars are driving directly towards each other such that one is twice as fast as the other. The distance between their starting points is 4 miles. When the two cars meet, how many miles is the faster car from its starting point?

2. [**4**] You are standing at a pole and a snail is moving directly away from the pole at 1 cm/s. When the snail is 1 meter away, you start "Round 1". In Round n ($n \geq 1$), you move directly toward the snail at $n+1$ cm/s. When you reach the snail, you immediately turn around and move back to the starting pole at $n+1$ cm/s. When you reach the pole, you immediately turn around and Round $n+1$ begins.

   At the start of Round 100, how many **meters** away is the snail?

3. [**5**] Let ABC be a triangle with $AB = 5$, $BC = 4$, and $CA = 3$. Initially, there is an ant at each vertex. The ants start walking at a rate of 1 unit per second, in the direction $A \to B \to C \to A$ (so the ant starting at A moves along ray $\overrightarrow{AB}$, etc.). For a positive real number t less than 3, let $A(t)$ be the area of the triangle whose vertices are the positions of the ants after t seconds have elapsed. For what positive real number t less than 3 is $A(t)$ minimized?

4. [**7**] There are 2 runners on the perimeter of a regular hexagon, initially located at adjacent vertices. Every second, each of the runners independently moves either one vertex to the left, with probability $\frac{1}{2}$, or one vertex to the right, also with probability $\frac{1}{2}$. Find the probability that after a 2013 second run (in which the runners switch vertices 2013 times each), the runners end up at adjacent vertices once again.

5. [**7**] Let ABC be a triangle with $AB = 13$, $BC = 14$, $CA = 15$. Company XYZ wants to locate their base at the point P in the plane minimizing the total distance to their workers, who are located at vertices A, B, and C. There are 1, 5, and 4 workers at A, B, and C, respectively. Find the minimum possible total distance Company XYZ's workers have to travel to get to P.

Bases

Many of you may be familiar with the decimal (or base 10) system. For example, when we say 2013_{10}, we really mean $2\cdot10^3+0\cdot10^2+1\cdot10^1+3\cdot10^0$. Similarly, there is the binary (base 2) system. For example, $11111011101_2 = 1\cdot2^{10}+1\cdot2^9+1\cdot2^8+1\cdot2^7+1\cdot2^6+0\cdot2^5+1\cdot2^4+1\cdot2^3+1\cdot2^2+0\cdot2^1+1\cdot2^0 = 2013_{10}$.

In general, if we are given a string $(a_n a_{n-1} \dots a_0)_b$ in base b (the subscript b means that we are in base b), then it is equal to $\sum_{i=0}^{n} a_i b^i$.

It turns out that for every positive integer $b > 1$, every positive integer k has a unique base b representation. That is, for every positive integer k, there exists a unique n and digits $0 \leq a_0, \dots, a_n < b$ such that $(a_n a_{n-1} \dots a_0)_b = k$.

We can adapt this to bases $b < -1$. It actually turns out that if $b < -1$, every *nonzero* integer has a unique base b representation. That is, for every nonzero integer k, there exists a unique n and digits $0 \leq a_0, \dots, a_n < |b|$ such that $(a_n a_{n-1} \dots a_0)_b = k$. The next five problems involve base -4.

Note: Unless otherwise stated, express your answers in base 10.

6. [**2**] Evaluate 1201201_{-4}.

7. [**3**] Express -2013 in base -4.

8. [**5**] Let $b(n)$ be the number of digits in the base -4 representation of n. Evaluate $\sum_{i=1}^{2013} b(i)$.

9. [**7**] Let N be the largest positive integer that can be expressed as a 2013-digit base -4 number. What is the remainder when N is divided by 210?

10. [**8**] Find the sum of all positive integers n such that there exists an integer b with $|b| \neq 4$ such that the base -4 representation of n is the same as the base b representation of n.