HMMT November 2013

Saturday 9 November 2013

Theme Round

1. [**2**] Two cars are driving directly towards each other such that one is twice as fast as the other. The distance between their starting points is 4 miles. When the two cars meet, how many miles is the faster car from its starting point?

   Answer: $\boxed{\frac{8}{3}}$ Note that the faster car traveled twice the distance of the slower car, and together, the two cars traveled the total distance between the starting points, which is 4 miles. Let the distance that the faster car traveled be x. Then, $x + \frac{x}{2} = 4 \implies x = \frac{8}{3}$. Thus, the faster car traveled $\frac{8}{3}$ miles from the starting point.

2. [**4**] You are standing at a pole and a snail is moving directly away from the pole at 1 cm/s. When the snail is 1 meter away, you start "Round 1". In Round n ($n \geq 1$), you move directly toward the snail at $n+1$ cm/s. When you reach the snail, you immediately turn around and move back to the starting pole at $n+1$ cm/s. When you reach the pole, you immediately turn around and Round $n+1$ begins.

   At the start of Round 100, how many **meters** away is the snail?

   Answer: $\boxed{5050}$ Suppose the snail is x_n meters away at the start of round n, so $x_1 = 1$, and the runner takes $\frac{100x_n}{(n+1)-1} = \frac{100x_n}{n}$ seconds to catch up to the snail. But the runner takes the same amount of time to run back to the start, so during round n, the snail moves a distance of $x_{n+1} - x_n = \frac{200x_n}{n}\frac{1}{100} = \frac{2x_n}{n}$.

   Finally, we have $x_{100} = \frac{101}{99}x_{99} = \frac{101}{99}\frac{100}{98}x_{98} = \cdots = \frac{101!/2!}{99!}x_1 = 5050$.

3. [**5**] Let ABC be a triangle with $AB = 5$, $BC = 4$, and $CA = 3$. Initially, there is an ant at each vertex. The ants start walking at a rate of 1 unit per second, in the direction $A \to B \to C \to A$ (so the ant starting at A moves along ray $\overrightarrow{AB}$, etc.). For a positive real number t less than 3, let $A(t)$ be the area of the triangle whose vertices are the positions of the ants after t seconds have elapsed. For what positive real number t less than 3 is $A(t)$ minimized?

   Answer: $\boxed{\frac{47}{24}}$ We instead maximize the area of the remaining triangles. This area (using $\frac{1}{2}xy\sin\theta$) is $\frac{1}{2}(t)(5-t)\frac{3}{5} + \frac{1}{2}(t)(3-t)\frac{4}{5} + \frac{1}{2}(t)(4-t)1 = \frac{1}{10}(-12t^2 + 47t)$, which has a maximum at $t = \frac{47}{24} \in (0,3)$.

4. [**7**] There are 2 runners on the perimeter of a regular hexagon, initially located at adjacent vertices. Every second, each of the runners independently moves either one vertex to the left, with probability $\frac{1}{2}$, or one vertex to the right, also with probability $\frac{1}{2}$. Find the probability that after a 2013 second run (in which the runners switch vertices 2013 times each), the runners end up at adjacent vertices once again.

   Answer: $\boxed{\frac{2}{3} + \frac{1}{3}(\frac{1}{4})^{2013} \text{ OR } \frac{2^{4027}+1}{3\cdot 2^{4026}} \text{ OR } \frac{2}{3} + \frac{1}{3}(\frac{1}{2})^{4026} \text{ OR } \frac{2}{3} + \frac{1}{3}(\frac{1}{64})^{671}}$ Label the runners A and B and arbitrarily fix an orientation of the hexagon. Let $p_t(i)$ be the probability that A is $i \pmod 6$ vertices to the right of B at time t, so without loss of generality $p_0(1) = 1$ and $p_0(2) = \cdots = p_0(6) = 0$. Then for $t > 0$, $p_t(i) = \frac{1}{4}p_{t-1}(i-2) + \frac{1}{2}p_{t-1}(i) + \frac{1}{4}p_{t-1}(i+2)$.

   In particular, $p_t(2) = p_t(4) = p_t(6) = 0$ for all t, so we may restrict our attention to $p_t(1), p_t(3), p_t(5)$. Thus $p_t(1) + p_t(3) + p_t(5) = 1$ for all $t \geq 0$, and we deduce $p_t(i) = \frac{1}{4} + \frac{1}{4}p_{t-1}(i)$ for $i = 1, 3, 5$.

   Finally, let $f(t) = p_t(1) + p_t(5)$ denote the probability that A, B are 1 vertex apart at time t, so $f(t) = \frac{1}{2} + \frac{1}{4}f(t-1) \implies f(t) - \frac{2}{3} = \frac{1}{4}[f(t-1) - \frac{2}{3}]$, and we conclude that $f(2013) = \frac{2}{3} + \frac{1}{3}(\frac{1}{4})^{2013}$.

5. [**7**] Let ABC be a triangle with $AB = 13$, $BC = 14$, $CA = 15$. Company XYZ wants to locate their base at the point P in the plane minimizing the total distance to their workers, who are located at vertices A, B, and C. There are 1, 5, and 4 workers at A, B, and C, respectively. Find the minimum possible total distance Company XYZ's workers have to travel to get to P.

   Answer: $\boxed{69}$ We want to minimize $1 \cdot PA + 5 \cdot PB + 4 \cdot PC$. By the triangle inequality, $(PA + PB) + 4(PB + PC) \geq AB + 4BC = 13 + 56 = 69$, with equality precisely when $P = [AB] \cap [BC] = B$.

6. [**2**] Evaluate 1201201_{-4}.

Answer: $\boxed{2017}$ The answer is $1+2(-4)^2+(-4)^3+2(-4)^5+(-4)^6 = 1-2\cdot4^2+2\cdot4^5 = 2049-32 = 2017$.

7. [**3**] Express -2013 in base -4.

Answer: $\boxed{200203}$ $-2013 \equiv 3 \pmod 4$, so the last digit is 3; now $\frac{-2013-3}{-4} = 504 \equiv 0$, so the next digit (to the left) is 0; then $\frac{504-0}{-4} = -126 \equiv 2$; $\frac{-126-2}{-4} = 32 \equiv 0$; $\frac{32-0}{-4} = -8 \equiv 0$; $\frac{-8-0}{-4} = 2$.

Thus $-2013_{10} = 200203_{-4}$.

8. [**5**] Let $b(n)$ be the number of digits in the base -4 representation of n. Evaluate $\sum_{i=1}^{2013} b(i)$.

Answer: $\boxed{12345}$ We have the following:

- $b(n) = 1$ for n between 1 and 3.
- $b(n) = 3$ for n between $4^2 - 3\cdot4 = 4$ and $3\cdot4^2+3 = 51$. (Since $a\cdot4^2 - b\cdot4 + c$ takes on $3\cdot4\cdot4$ distinct values over $1 \le a \le 3$, $0 \le b \le 3$, $0 \le c \le 3$, with minimum 4 and maximum 51.)
- $b(n) = 5$ for n between $4^4 - 3\cdot4^3 - 3\cdot4 = 52$ and $3\cdot4^4 + 3\cdot4^2 + 3 = 819$.
- $b(n) = 7$ for n between $4^6 - 3\cdot4^5 - 3\cdot4^3 - 3\cdot4^1 = 820$ and $3\cdot4^6 + 3\cdot4^4 + 3\cdot4^2 + 3 > 2013$.

Thus

$$\sum_{i=1}^{2013} b(i) = 7(2013) - 2(819+51+3) = 14091 - 2(873) = 14091 - 1746 = 12345.$$

9. [**7**] Let N be the largest positive integer that can be expressed as a 2013-digit base -4 number. What is the remainder when N is divided by 210?

Answer: $\boxed{51}$ The largest is $\sum_{i=0}^{1006} 3\cdot4^{2i} = 3\frac{16^{1007}-1}{16-1} = \frac{16^{1007}-1}{5}$.

This is $1 \pmod 2$, $0 \pmod 3$, $3\cdot1007 \equiv 21 \equiv 1 \pmod 5$, and $3(2^{1007}-1) \equiv 3(2^8-1) \equiv 3(2^2-1) \equiv 2 \pmod 7$, so we need $1 \pmod{10}$ and $9 \pmod{21}$, which is $9 + 2\cdot21 = 51 \pmod{210}$.

10. [**8**] Find the sum of all positive integers n such that there exists an integer b with $|b| \ne 4$ such that the base -4 representation of n is the same as the base b representation of n.

Answer: $\boxed{1026}$ All 1 digit numbers, $0,1,2,3$, are solutions when, say, $b = 5$. (Of course, $d \in \{0,1,2,3\}$ works for any base b of absolute value greater than d but not equal to 4.)

Consider now positive integers $n = (a_d \dots a_1 a_0)_4$ with more than one digit, so $d \ge 1$, $a_d \ne 0$, and $0 \le a_k \le 3$ for $k = 0,1,\dots,d$. Then n has the same representation in base b if and only if $|b| > \max a_k$ and $\sum_{k=0}^d a_k(-4)^k = \sum_{k=0}^d a_k b^k$, or equivalently, $\sum_{k=0}^d a_k(b^k - (-4)^k) = 0$.

First we prove that $b \le 3$. Indeed, if $b \ge 4$, then $b \ne 4 \implies b \ge 5$, so $b^k - (-4)^k$ is positive for all $k \ge 1$ (and zero for $k = 0$). But then $\sum_{k=0}^d a_k(b^k - (-4)^k) \ge a_d(b^d - (-4)^d)$ must be positive, and cannot vanish.

Next, we show $b \ge 2$. Assume otherwise for the sake of contradiction; b cannot be $0, \pm1$ (these bases don't make sense in general) or -4, so we may label two distinct negative integers $-r, -s$ with $r - 1 \ge s \ge 2$ such that $\{r, s\} = \{4, -b\}$, $s > \max a_k$, and $\sum_{k=0}^d a_k((-r)^k - (-s)^k) = 0$, which, combined with the fact that $r^k - s^k \ge 0$ (equality only at $k = 0$), yields

$$\begin{aligned} r^d - s^d \le a_d(r^d - s^d) &= \sum_{k=0}^{d-1} (-1)^{d-1-k} a_k (r^k - s^k) \\ &\le \sum_{k=0}^{d-1} (s-1)(r^k - s^k) = (s-1)\frac{r^d-1}{r-1} - (s^d - 1). \end{aligned}$$

Hence $r^d - 1 \le (s-1)\frac{r^d-1}{r-1} < (r-1)\frac{r^d-1}{r-1} = r^d - 1$, which is absurd.

Thus $b \geq 2$, and since $b \leq 3$ we must either have $b = 2$ or $b = 3$. In particular, all a_k must be at most $b - 1$. We now rewrite our condition as

$$a_d(4^d - (-b)^d) = \sum_{k=0}^{d-1} (-1)^{d-1-k} a_k (4^k - (-b)^k).$$

Since $4^k - (-b)^k \geq 0$ for $k \geq 0$, with equality only at $k = 0$, we deduce

$$a_d(4^d - (-b)^d) \leq \sum_{k \equiv d-1 \pmod 2} (b-1)(4^k - (-b)^k).$$

If $d - 1$ is even (d is odd), this gives

$$a_d(4^d + b^d) \leq (b-1)\frac{4^{d+1} - 4^0}{4^2 - 1} - (b-1)\frac{b^{d+1} - b^0}{b^2 - 1},$$

so $4^d < (b-1)\frac{4^{d+1}}{15} \implies b > 1 + \frac{15}{4}$, which is impossible.
Thus $d - 1$ is odd (d is even), and we get

$$a_d(4^d - b^d) \leq (b-1)\frac{4^{d+1} - 4^1}{4^2 - 1} + (b-1)\frac{b^{d+1} - b^1}{b^2 - 1} \iff \frac{b^d - 1}{4^d - 1} \geq \frac{a_d - \frac{4}{15}(b-1)}{a_d + \frac{b}{b+1}}.$$

If $b = 2$, then $a_d = 1$, so $\frac{1}{2^d+1} = \frac{2^d-1}{4^d-1} \geq \frac{11}{25}$, which is clearly impossible ($d \geq 2$).

If $b = 3$ and $a_d = 2$, then $\frac{9^{d/2}-1}{16^{d/2}-1} \leq \frac{8}{15}$. Since d is even, it's easy to check this holds only for $d/2 = 1$, with equality, so $a_k = b - 1$ if $k \equiv d - 1 \pmod 2$. Thus $(a_d, \ldots, a_0) = (2, 2, a_0)$, yielding solutions $(22x)_3$ (which do work; note that the last digit doesn't matter).

Otherwise, if $b = 3$ and $a_d = 14$, then $\frac{9^{d/2}-1}{16^{d/2}-1} \leq \frac{4}{15}$. It's easy to check $d/2 \in \{1, 2\}$.

If $d/2 = 1$, we're solving $16a_2 - 4a_1 + a_0 = 9a_2 + 3a_1 + a_0 \iff a_2 = a_1$. We thus obtain the working solution $(11x)_3$. (Note that $110 = \frac{1}{2}220$ in bases $-4, 3$.)

If $d/2 = 2$, we want $256a_4 - 64a_3 + 16a_2 - 4a_1 + a_0 = 81a_4 + 27a_3 + 9a_2 + 3a_1 + a_0$, or $175 = 91a_3 - 7a_2 + 7a_1$, which simplifies to $25 = 13a_3 - a_2 + a_1$. This gives the working solutions $(1210x)_3$, $(1221x)_3$. (Note that $12100 = 110^2$ and $12210 = 110^2 + 110$ in bases $-4, 3$.)

The list of all nontrivial ($\geq$ 2-digit) solutions (in base -4 and b) is then $11x$, $22x$, $1210x$, $1221x$, where $b = 3$ and $x \in \{0, 1, 2\}$. In base 10, they are $12 + x$, $2 \cdot 12 + x$, $12^2 + x$, $12^2 + 12 + x$, with sum $3(2 \cdot 12^2 + 4 \cdot 12) + 4(0 + 1 + 2) = 1020$.

Finally, we need to include the trivial solutions $n = 1, 2, 3$, for a total sum of 1026.