HMMT 2014

Saturday 22 February 2014

Combinatorics

1. There are 100 students who want to sign up for the class Introduction to Acting. There are three class sections for Introduction to Acting, each of which will fit exactly 20 students. The 100 students, including Alex and Zhu, are put in a lottery, and 60 of them are randomly selected to fill up the classes. What is the probability that Alex and Zhu end up getting into the same section for the class?

   Answer: $\boxed{19/165}$ There is a $\frac{60}{100} = \frac{3}{5}$ chance that Alex is in the class. If Alex is in the class, the probability that Zhu is in his section is $\frac{19}{99}$. So the answer is $\frac{3}{5} \cdot \frac{19}{99} = \frac{19}{165}$.

2. There are 10 people who want to choose a committee of 5 people among them. They do this by first electing a set of 1, 2, 3, or 4 committee leaders, who then choose among the remaining people to complete the 5-person committee. In how many ways can the committee be formed, assuming that people are distinguishable? (Two committees that have the same members but different sets of leaders are considered to be distinct.)

   Answer: $\boxed{7560}$ There are $\binom{10}{5}$ ways to choose the 5-person committee. After choosing the committee, there are $2^5 - 2 = 30$ ways to choose the leaders. So the answer is $30 \cdot \binom{10}{5} = 7560$.

3. Bob writes a random string of 5 letters, where each letter is either A, B, C, or D. The letter in each position is independently chosen, and each of the letters A, B, C, D is chosen with equal probability. Given that there are at least two A's in the string, find the probability that there are at least three A's in the string.

   Answer: $\boxed{\frac{53}{188}}$ There are $\binom{5}{2}3^3 = 270$ strings with 2 A's. There are $\binom{5}{3}3^2 = 90$ strings with 3 A's. There are $\binom{5}{4}3^1 = 15$ strings with 4 A's. There is $\binom{5}{5}3^0 = 1$ string with 5 A's.
   The desired probability is $\frac{90+15+1}{270+90+15+1} = \frac{53}{188}$.

4. Find the number of triples of sets (A, B, C) such that:

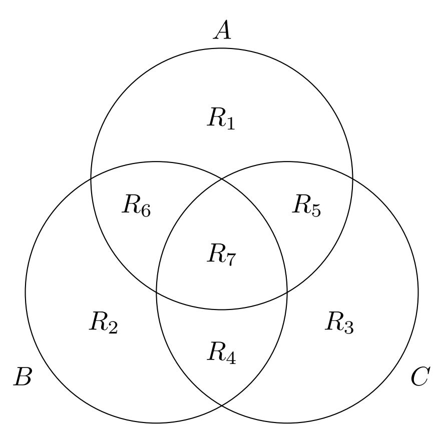
   (a) $A, B, C \subseteq \{1, 2, 3, \ldots, 8\}$.

   (b) $|A \cap B| = |B \cap C| = |C \cap A| = 2$.

   (c) $|A| = |B| = |C| = 4$.

   Here, $|S|$ denotes the number of elements in the set S.

   Answer: $\boxed{45360}$ We consider the sets drawn in a Venn diagram.



Note that each element that is in at least one of the subsets lies in these seven possible spaces. We split by casework, with the cases based on $N = |R_7| = |A \cap B \cap C|$.

Case 1: $N = 2$

Because we are given that $|R_4| + N = |R_5| + N = |R_6| + N = 2$, we must have $|R_4| = |R_5| = |R_6| = 0$. But we also know that $|R_1| + |R_5| + |R_6| + N = 4$, so $|R_1| = 2$. Similarly, $|R_2| = |R_3| = 2$. Since these regions are distinguishable, we multiply through and obtain $\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = 2520$ ways.

Case 2: $N = 1$

In this case, we can immediately deduce $|R_4| = |R_5| = |R_6| = 1$. From this, it follows that $|R_1| = 4 - 1 - 1 - 1 = 1$, and similarly, $|R_2| = |R_3| = 1$. All seven regions each contain one integer, so there are a total of $(8)(7)\dots(2) = 40320$ ways.

Case 3: $N = 0$

Because $|R_4| + N = |R_5| + N = |R_6| + N = 2$, we must have $|R_4| = |R_5| = |R_6| = 2$. Since $|R_1| + |R_5| + |R_6| + N = 4$, we immediately see that $|R_1| = 0$. Similarly, $|R_2| = |R_3| = 0$. The number of ways to fill R_4, R_5, R_6 is $\binom{8}{2}\binom{6}{2}\binom{4}{2} = 2520$.

This clearly exhausts all the possibilities, so adding gives us $40320 + 2520 + 2520 = 45360$ ways.

5. Eli, Joy, Paul, and Sam want to form a company; the company will have 16 shares to split among the 4 people. The following constraints are imposed:

   - Every person must get a positive integer number of shares, and all 16 shares must be given out.
   - No one person can have more shares than the other three people combined.

   Assuming that shares are indistinguishable, but people are distinguishable, in how many ways can the shares be given out?

   Answer: $\boxed{315}$ We are finding the number of integer solutions to $a + b + c + d = 16$ with $1 \leq a, b, c, d \leq 8$. We count the number of solutions to $a+b+c+d = 16$ over positive integers, and subtract the number of solutions in which at least one variable is larger than 8. If at least one variable is larger than 8, exactly one of the variables is larger than 8. We have 4 choices for this variable. The number of solutions to $a + b + c + d = 16$ over positive integers, where $a > 8$, is just the number of solutions to $a' + b + c + d = 8$ over positive integers, since we can substitute $a' = a - 8$. Thus, by the stars and bars formula (the number of positive integer solutions to $x_1 + \cdots + x_m = n$ is $\binom{n-1}{m-1}$), the answer is $\binom{16-1}{4-1} - \binom{4}{1}\binom{(16-8)-1}{4-1} = 35 \cdot 13 - 4 \cdot 35 = 315$.

6. We have a calculator with two buttons that displays an integer x. Pressing the first button replaces x by $\lfloor \frac{x}{2} \rfloor$, and pressing the second button replaces x by $4x + 1$. Initially, the calculator displays 0. How many integers less than or equal to 2014 can be achieved through a sequence of arbitrary button presses? (It is permitted for the number displayed to exceed 2014 during the sequence. Here, $\lfloor y \rfloor$ denotes the greatest integer less than or equal to the real number y.)

   Answer: $\boxed{233}$ We consider the integers from this process written in binary. The first operation truncates the rightmost digit, while the second operation appends 01 to the right.

   We cannot have a number with a substring 11. For simplicity, call a string *valid* if it has no consecutive $1's$. Note that any number generated by this process is valid, as truncating the rightmost digit and appending 01 to the right of the digits clearly preserve validity.

   Since we can effectively append a zero by applying the second operation and then the first operation, we see that we can achieve all valid strings.

Note that 2014 has eleven digits when written in binary, and any valid binary string with eleven digits is at most $10111111111 = 1535$. Therefore, our problem reduces to finding the number of eleven-digit valid strings. Let F_n denote the number of valid strings of length n. For any valid string of length n, we can create a valid string of length $n+1$ by appending a 0, or we can create a valid string of length $n+2$ by appending 01. This process is clearly reversible, so our recursion is given by $F_n = F_{n-1} + F_{n-2}$, with $F_1 = 2, F_2 = 3$. This yields a sequence of Fibonacci numbers starting from 2, and some computation shows that our answer is $F_{11} = 233$.

7. Six distinguishable players are participating in a tennis tournament. Each player plays one match of tennis against every other player. There are no ties in this tournament—each tennis match results in a win for one player and a loss for the other. Suppose that whenever A and B are players in the tournament such that A wins strictly more matches than B over the course of the tournament, it is also true that A wins the match against B in the tournament. In how many ways could the tournament have gone?

   Answer: $\boxed{2048}$ We first group the players by wins, so let G_1 be the set of all players with the most wins, G_2 be the set of all players with the second most wins, ..., G_n be the set of all players with the least wins. By the condition in the problem, everyone in group G_i must beat everyone in group G_j for all $i < j$. Now, consider the mini-tournament consisting of the matches among players inside a single group G_i. Each must have the same number of wins, say x_i. But the total number of games is $\binom{|G_i|}{2}$ and each game corresponds to exactly one win, so we must have $\binom{|G_i|}{2} = |G_i|x_i \implies |G_i| = 2x_i + 1$. Therefore, the number of players in each G_i is odd.

   We now have $\sum |G_i| = 6$ and all $|G_i|$ are odd, so we can now do casework on the possibilities.

   Case 1: G_i's have sizes 5 and 1. In this case, there are 2 ways to permute the groups (i.e. either $|G_1| = 5, |G_2| = 1$ or $|G_1| = 1, |G_2| = 5$). There are 6 ways to distribute the players into the two groups. There are 24 possible mini-tournaments in the group of size 5; to prove this, we label the players $p_1, \ldots, p_5$ and note that each player has 2 wins. Without loss of generality, let p_1 beat p_2 and p_3, and also without loss of generality let p_2 beat p_3. It's easy to verify that there are 2 possible mini-tournaments, depending on whether p_4 beats p_5 or p_5 beats p_4. Since there are $\binom{4}{2} \cdot 2 = 12$ ways to pick the two players p_1 defeats and choose which one beats the other, there are indeed $12 \cdot 2 = 24$ tournaments. Then the total number of possible tournaments in this case is $2 \cdot 6 \cdot 24 = 288$.

   Case 2: The sizes are 3, 3. In this case, there are $\binom{6}{3} = 20$ ways to distribute the players into the groups, and 2 possible mini-tournaments in either group, so the total here is $20 \cdot 2 \cdot 2 = 80$.

   Case 3: The sizes are 3, 1, 1, 1. In this case, there are 4 ways to permute the groups, $\binom{6}{3} \cdot 6 = 120$ ways to distribute the players into groups, and 2 possible mini-tournaments in the group of size 3, for a total of $4 \cdot 120 \cdot 2 = 960$.

   Case 4: The sizes are 1, 1, 1, 1, 1, 1. There are 720 ways to distribute the players into groups.

   The final answer is $288 + 80 + 960 + 720 = 2048$.

8. The integers $1, 2, \ldots, 64$ are written in the squares of a 8×8 chess board, such that for each $1 \le i < 64$, the numbers i and $i+1$ are in squares that share an edge. What is the largest possible sum that can appear along one of the diagonals?

   Answer: $\boxed{432}$ Our answer is 26 + 52 + 54 + 56 + 58 + 60 + 62 + 64.
   One possible configuration:

   WLOG, we seek to maximize the sum of the numbers on the main diagonal (top left to bottom right). If we color the squares in a checker-board pattern and use the fact that a and $a+1$ lie on different colored squares, we notice that all numbers appearing on the main diagonal must be of the same parity.

   Consider the smallest value m on the main diagonal. All numbers from 1 to $m-1$ must lie on one side of the diagonal since the main diagonal disconnects the board into two regions, and by assumption, all

26	25	24	23	18	17	8	7
27	52	53	22	19	16	9	6
28	51	54	21	20	15	10	5
29	50	55	56	57	14	11	4
30	49	44	43	58	13	12	3
31	48	45	42	59	60	61	2
32	47	46	41	40	39	62	1
33	34	35	36	37	38	63	64

numbers less than m cannot lie on the main diagonal. Therefore, $m \leq 29$ (one more than the seventh triangular number) But if $m = 29$, then the sum of the numbers on the main diagonal is at most $29 + 51 + 53 + 55 + 57 + 59 + 61 + 63 = 428$, as these numbers must be odd. Similarly, $m = 27$ is also not optimal.

This leaves $m = 28$ as a possibility. But if this were the case, the only way it beats our answer is if we have $28 + 52 + 54 + ... + 64$, which would require $52, 54, ..., 64$ to appear sequentially along the diagonal, forcing 28 to be in one of the corners.

Now label the squares $(row, column)$ with $(1, 1)$ being the top left and $(8, 8)$ being the bottom right. Assume WLOG that 28 occupies $(1, 1)$. Since 62 and 64 are in (7, 7) and (8, 8), respectively, we must have 63 in (7, 8) or (8, 7), and WLOG, assume it's in (8, 7). Since 61 is next to 60, it is not difficult to see that (7, 8) must be occupied by 1 (all numbers a between 2 and 60 must have $a - 1$ and $a + 1$ as neighbors) . Since 1 is above the main diagonal, all numbers from 1 to 27 must also be above the main diagonal. Since there are 28 squares above the main diagonal, there is exactly one number above the main diagonal greater than 28.

Notice that 61 must occupy $(7, 6)$ or $(6, 7)$. If it occupies $(7, 6)$, then we are stuck at $(8, 6)$, since it must contain a number between 2 and 59, which is impossible. Therefore, 61 must occupy $(6, 7)$, and no more numbers greater than 28 can be above the main diagonal. This forces 59, 57, 55, and 53 to occupy $(6, 5), (5, 4), (4, 3), (3, 2)$, respectively. But we see that 27 occupies $(1, 2)$ and 29 occupies $(2, 1)$, leaving nowhere for 51.

This is a contradiction, so our answer is therefore optimal.

Alternate solution: Another method of proving that $m \leq 26$ is to note that each side of the diagonal has 28 squares, 16 of which are one color and 12 of which are the other color. As the path has to alternate colors, one can make at most $13 + 12 = 25$ steps before moving on the diagonal.

9. There is a heads up coin on every integer of the number line. Lucky is initially standing on the zero point of the number line facing in the positive direction. Lucky performs the following procedure: he looks at the coin (or lack thereof) underneath him, and then,

   - If the coin is heads up, Lucky flips it to tails up, turns around, and steps forward a distance of one unit.
   - If the coin is tails up, Lucky picks up the coin and steps forward a distance of one unit facing the same direction.
   - If there is no coin, Lucky places a coin heads up underneath him and steps forward a distance of one unit facing the same direction.

   He repeats this procedure until there are 20 coins anywhere that are tails up. How many times has Lucky performed the procedure when the process stops?

Answer: $\boxed{6098}$ We keep track of the following quantities: Let N be the sum of 2^k, where k ranges over all nonnegative integers such that position $-1-k$ on the number line contains a tails-up coin. Let M be the sum of 2^k, where k ranges over all nonnegative integers such that position k contains a tails-up coin.

We also make the following definitions: A "right event" is the event that Lucky crosses from the negative integers on the number line to the non-negative integers. A "left event" is the event that Lucky crosses from the non-negative integers on the number line to the negative integers.

We now make the following claims:

(a) Every time a right event or left event occurs, every point on the number line contains a coin.

(b) Suppose that n is a positive integer. When the nth left event occurs, the value of M is equal to n. When the nth right event occurs, the value of N is equal to n.

(c) For a nonzero integer n, denote by $\nu_2(n)$ the largest integer k such that 2^k divides n. The number of steps that elapse between the $(n-1)$st right event and the nth left event is equal to $2\nu_2(n)+1$. The number of steps that elapse between the nth left event and the nth right event is also equal to $2\nu_2(n)+1$. (If $n-1=0$, then the "$(n-1)$st right event" refers to the beginning of the simulation.)

(d) The man stops as soon as the 1023rd right event occurs. (Note that $1023 = 2^{10}-1$.)

In other words, Lucky is keeping track of two numbers M and N, which are obtained by interpreting the coins on the number line as binary strings, and alternately incrementing each of them by one. We will prove claim 2; the other claims follow from very similar reasoning and their proofs will be omitted.

Clearly, left and right events alternate. That is, a left event occurs, then a right event, then a left event, and so on. So it's enough to prove that, between each right event and the following left event, the value of M is incremented by 1, and that between each left event and the following right event, the value of N is incremented by 1. We will show the first statement; the second follows from symmetry.

Suppose that a right event has just occurred. Then, by claim 1, every space on the number line cotnains a coin. So, there is some nonnegative integer ℓ for which positions $0, \ldots, \ell-1$ on the number line contain a tails up coin, and position ℓ contains a heads up coin. Since Lucky is standing at position 0 facing rightward, the following sequence of steps will occur:

(a) Lucky will take ℓ steps to the right, eventually reaching position ℓ. During this process, he will pick up the coins at positions $0, \ldots, \ell-1$.

(b) Then, Lucky turn the coin at position ℓ to a tails up coin and turn around.

(c) Finally, Lucky will take $\ell+1$ steps to the left, eventually reaching position -1 (at which point a left event occurs). During this process, he will place a heads up coin at positions $0, \ldots, \ell-1$.

During this sequence, the tails up coins at positions $0, \ldots, \ell-1$ have been changed to heads up coins, and the heads up coin at position ℓ has been changed to a tails up coin. So the value of M has been incremented by

$$2^\ell - \sum_{i=0}^{\ell-1} 2^i = 1$$

as desired.

Now, it remains to compute the answer to the question. By claims 3 and 4, the total number of steps taken by the simulation is

$$2\sum_{n=1}^{1023}(2\nu_2(n)+1).$$

This can be rewritten as

$$4\sum_{n=1}^{1023}\nu_2(n) + 2\cdot 1023 = 4\nu_2(1023!) + 2046.$$

We can compute $\nu_2(1023!) = 1013$ using Legendre's formula for the highest power of 2 dividing a factorial. This results in the final answer 6098.

10. An *up-right path* from $(a, b) \in \mathbb{R}^2$ to $(c, d) \in \mathbb{R}^2$ is a finite sequence $(x_1, y_1), \ldots, (x_k, y_k)$ of points in $\mathbb{R}^2$ such that $(a, b) = (x_1, y_1)$, $(c, d) = (x_k, y_k)$, and for each $1 \leq i < k$ we have that either $(x_{i+1}, y_{i+1}) = (x_i + 1, y_i)$ or $(x_{i+1}, y_{i+1}) = (x_i, y_i + 1)$. Two up-right paths are said to intersect if they share any point.

    Find the number of pairs (A, B) where A is an up-right path from $(0, 0)$ to $(4, 4)$, B is an up-right path from $(2, 0)$ to $(6, 4)$, and A and B do not intersect.

    Answer: $\boxed{1750}$ The number of up-right paths from $(0, 0)$ to $(4, 4)$ is $\binom{8}{4}$ because any such up-right path is identical to a sequence of 4 U's and 4 R's, where U corresponds to a step upwards and R corresponds to a step rightwards. Therefore, the total number of pairs of (possibly intersecting) up-right paths from $(0, 0)$ to $(4, 4)$ and $(2, 0)$ to $(6, 4)$ is $\binom{8}{4}^2$.

    We will now count the number of intersecting pairs of up-right paths and subtract it to get the answer. Consider an up-right path A from $(0, 0)$ to $(4, 4)$ and an up-right path B from $(2, 0)$ to $(6, 4)$. If they intersect, take the point (x, y) where they first meet each other, and switch the parts of the paths after (x, y) to make an up-right path A' from $(0, 0)$ to $(6, 4)$ and an up-right path B' from $(2, 0)$ to $(4, 4)$.

    Conversely, given an up-right path A' from $(0, 0)$ to $(6, 4)$ and an up-right path B' from $(2, 0)$ to $(4, 4)$, they must intersect somewhere, so we can again take their first intersection point and switch the ends to get the original up-right path A from $(0, 0)$ to $(4, 4)$ and up-right path B from $(2, 0)$ to $(6, 4)$, where A and B intersect.

    Consequently, the number of intersecting pairs of up-right paths is exactly equal to the number of pairs of up-right paths from $(0, 0)$ to $(6, 4)$ and $(2, 0)$ to $(4, 4)$, which is $\binom{10}{4}\binom{6}{4}$. The number of pairs that do not intersect is therefore $\binom{8}{4}^2 - \binom{10}{4}\binom{6}{4} = 4900 - 3150 = 1750$.