HMMT November 2014

Saturday 15 November 2014

Theme Round

Townspeople and Goons

In the city of Lincoln, there is an empty jail, at least two *townspeople* and at least one *goon*. A game proceeds over several days, starting with morning.

- Each morning, one randomly selected unjailed person is placed in jail. If at this point all goons are jailed, and at least one townsperson remains, then the townspeople win. If at this point all townspeople are jailed and at least one goon remains, then the goons win.
- Each evening, if there is at least one goon and at least one townsperson not in jail, then one randomly selected townsperson is jailed. If at this point there are at least as many goons remaining as townspeople remaining, then the goons win.

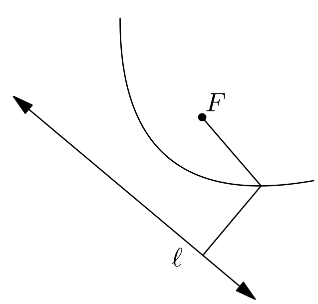
The game ends immediately after any group wins.

1. Find the probability that the townspeople win if there are initially two townspeople and one goon.
2. Find the smallest positive integer n such that, if there are initially $2n$ townspeople and 1 goon, then the probability the townspeople win is greater than 50%.
3. Find the smallest positive integer n such that, if there are initially $n+1$ townspeople and n goons, then the probability the townspeople win is less than 1%.
4. Suppose there are initially 1001 townspeople and two goons. What is the probability that, when the game ends, there are exactly 1000 people in jail?
5. Suppose that there are initially eight townspeople and one goon. One of the eight townspeople is named *Jester*. If Jester is sent to jail during some morning, then the game ends immediately in his sole victory. (However, the Jester does not win if he is sent to jail during some night.)

   Find the probability that only the Jester wins.

Parabolas

A *parabola* is the set of points in the plane equidistant from a point F (its *focus*) and a line ℓ (its directrix).



Parabolas can be represented as polynomial equations of degree two, such as $y = x^2$.

6. Let $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ be pairwise distinct parabolas in the plane. Find the maximum possible number of intersections between two or more of the $\mathcal{P}_i$. In other words, find the maximum number of points that can lie on two or more of the parabolas $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$.

7. Let $\mathcal{P}$ be a parabola with focus F and directrix ℓ. A line through F intersects $\mathcal{P}$ at two points A and B. Let D and C be the feet of the altitudes from A and B onto ℓ, respectively. Given that $AB = 20$ and $CD = 14$, compute the area of $ABCD$.

8. Consider the parabola consisting of the points (x, y) in the real plane satisfying

$$(y + x) = (y - x)^2 + 3(y - x) + 3.$$

   Find the minimum possible value of y.

9. In equilateral triangle ABC with side length 2, let the parabola with focus A and directrix BC intersect sides AB and AC at A_1 and A_2, respectively. Similarly, let the parabola with focus B and directrix CA intersect sides BC and BA at B_1 and B_2, respectively. Finally, let the parabola with focus C and directrix AB intersect sides CA and CB at C_1 and C_2, respectively.

   Find the perimeter of the triangle formed by lines A_1A_2, B_1B_2, C_1C_2.

10. Let z be a complex number and k a positive integer such that z^k is a positive real number other than 1. Let $f(n)$ denote the real part of the complex number z^n. Assume the parabola $p(n) = an^2 + bn + c$ intersects $f(n)$ four times, at $n = 0, 1, 2, 3$. Assuming the smallest possible value of k, find the largest possible value of a.