HMMT FEBRUARY 2015, 21 FEBRUARY 2015 — GUTS ROUND

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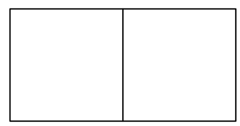
1. [**4**] Let R be the rectangle in the Cartesian plane with vertices at $(0,0),(2,0),(2,1)$, and $(0,1)$. R can be divided into two unit squares, as shown.



The resulting figure has 7 segments of unit length, connecting neighboring lattice points (those lying on or inside R). Compute the number of paths from $(0,1)$ (the upper left corner) to $(2,0)$ (the lower right corner) along these 7 segments, where each segment can be used at most once.

2. [**4**] Let $ABCDE$ be a convex pentagon such that $\angle ABC = \angle ACD = \angle ADE = 90^\circ$ and $AB = BC = CD = DE = 1$. Compute AE.

3. [**4**] Find the number of pairs of union/intersection operations $(\Box_1, \Box_2) \in \{\cup, \cap\}^2$ satisfying the following condition: for any sets S, T, function $f : S \to T$, and subsets X, Y, Z of S, we have equality of **sets**

$$f(X)\Box_1(f(Y)\Box_2 f(Z)) = f\left(X\Box_1(Y\Box_2 Z)\right),$$

where $f(X)$ denotes the *image* of X: the **set** $\{f(x) : x \in X\}$, which is a subset of T. The images $f(Y)$ (of Y) and $f(Z)$ (of Z) are similarly defined.

4. [**4**] Consider the function $z(x,y)$ describing the paraboloid

$$z = (2x-y)^2 - 2y^2 - 3y.$$

Archimedes and Brahmagupta are playing a game. Archimedes first chooses x. Afterwards, Brahmagupta chooses y. Archimedes wishes to minimize z while Brahmagupta wishes to maximize z. Assuming that Brahmagupta will play optimally, what value of x should Archimedes choose?

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5. [**5**] Let $\mathcal{H}$ be the unit hypercube of dimension 4 with a vertex at (x, y, z, w) for each choice of $x, y, z, w \in \{0, 1\}$. (Note that $\mathcal{H}$ has $2^4 = 16$ vertices.) A bug starts at the vertex $(0, 0, 0, 0)$. In how many ways can the bug move to $(1, 1, 1, 1)$ (the opposite corner of $\mathcal{H}$) by taking exactly 4 steps along the edges of $\mathcal{H}$?

6. [**5**] Let D be a regular ten-sided polygon with edges of length 1. A triangle T is defined by choosing three vertices of D and connecting them with edges. How many different (non-congruent) triangles T can be formed?

7. [**5**] Let $\mathcal{C}$ be a cube of side length 2. We color each of the faces of $\mathcal{C}$ blue, then subdivide it into $2^3 = 8$ unit cubes. We then randomly rearrange these cubes (possibly with rotation) to form a new 3-dimensional cube.

   What is the probability that its exterior is still completely blue?

8. [**5**] Evaluate
$$\sin(\arcsin(0.4) + \arcsin(0.5)) \cdot \sin(\arcsin(0.5) - \arcsin(0.4)),$$
   where for $x \in [-1, 1]$, $\arcsin(x)$ denotes the unique real number $y \in [-\pi, \pi]$ such that $\sin(y) = x$.

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9. [**6**] Let a, b, c be integers. Define $f(x) = ax^2 + bx + c$. Suppose there exist pairwise distinct integers u, v, w such that $f(u) = 0$, $f(v) = 0$, and $f(w) = 2$. Find the maximum possible value of the discriminant $b^2 - 4ac$ of f.

10. [**6**] Let $b(x) = x^2 + x + 1$. The polynomial $x^{2015} + x^{2014} + \cdots + x + 1$ has a unique "base $b(x)$" representation

$$x^{2015} + x^{2014} + \cdots + x + 1 = \sum_{k=0}^{N} a_k(x) b(x)^k,$$

where

- N is a nonnegative integer;
- each "digit" $a_k(x)$ (for $0 \leq k \leq N$) is either the zero polynomial (i.e. $a_k(x) = 0$) or a nonzero polynomial of degree less than $\deg b = 2$; and
- the "leading digit $a_N(x)$" is nonzero (i.e. not the zero polynomial).

Find $a_N(0)$ (the "leading digit evaluated at 0").

11. [**6**] Find

$$\sum_{k=0}^{\infty} \left\lfloor \frac{1 + \sqrt{\frac{2000000}{4^k}}}{2} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

12. [**6**] For integers a, b, c, d, let $f(a, b, c, d)$ denote the number of ordered pairs of integers $(x, y) \in \{1, 2, 3, 4, 5\}^2$ such that $ax + by$ and $cx + dy$ are both divisible by 5. Find the sum of all possible values of $f(a, b, c, d)$.

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13. [**8**] Let $P(x) = x^3 + ax^2 + bx + 2015$ be a polynomial all of whose roots are integers. Given that $P(x) \geq 0$ for all $x \geq 0$, find the sum of all possible values of $P(-1)$.

14. [**8**] Find the smallest integer $n \geq 5$ for which there exists a set of n distinct pairs $(x_1, y_1), \ldots, (x_n, y_n)$ of positive integers with $1 \leq x_i, y_i \leq 4$ for $i = 1, 2, \ldots, n$, such that for any indices $r, s \in \{1, 2, \ldots, n\}$ (not necessarily distinct), there exists an index $t \in \{1, 2, \ldots, n\}$ such that 4 divides $x_r + x_s - x_t$ and $y_r + y_s - y_t$.

15. [**8**] Find the maximum possible value of $H \cdot M \cdot M \cdot T$ over all ordered triples (H, M, T) of integers such that $H \cdot M \cdot M \cdot T = H + M + M + T$.

16. [**8**] Determine the number of **unordered** triples of **distinct** points in the $4 \times 4 \times 4$ lattice grid $\{0, 1, 2, 3\}^3$ that are collinear in $\mathbb{R}^3$ (i.e. there exists a line passing through the three points).

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17. [**11**] Find the least positive integer $N > 1$ satisfying the following two properties:

- There exists a positive integer a such that $N = a(2a - 1)$.
- The sum $1 + 2 + \cdots + (N - 1)$ is divisible by k for every integer $1 \le k \le 10$.

18. [**11**] Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function such that for any integers x, y, we have

$$f(x^2 - 3y^2) + f(x^2 + y^2) = 2(x + y)f(x - y).$$

Suppose that $f(n) > 0$ for all $n > 0$ and that $f(2015) \cdot f(2016)$ is a perfect square. Find the minimum possible value of $f(1) + f(2)$.

19. [**11**] Find the smallest positive integer n such that the polynomial $(x + 1)^n - 1$ is "divisible by $x^2 + 1$ modulo 3", or more precisely, either of the following **equivalent** conditions holds:

- there exist polynomials P, Q with integer coefficients such that $(x+1)^n - 1 = (x^2+1)P(x) + 3Q(x)$;
- or more conceptually, the remainder when (the polynomial) $(x + 1)^n - 1$ is divided by (the polynomial) $x^2 + 1$ is a polynomial with (integer) coefficients all divisible by 3.

20. [**11**] What is the largest real number θ less than π (i.e. $\theta < \pi$) such that

$$\prod_{k=0}^{10} \cos(2^k\theta) \neq 0$$

and

$$\prod_{k=0}^{10} \left(1 + \frac{1}{\cos(2^k\theta)}\right) = 1?$$

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21. [**14**] Define a sequence $a_{i,j}$ of integers such that $a_{1,n} = n^n$ for $n \geq 1$ and $a_{i,j} = a_{i-1,j} + a_{i-1,j+1}$ for all $i, j \geq 1$. Find the last (decimal) digit of $a_{128,1}$.

22. [**14**] Let $A_1, A_2, \ldots, A_{2015}$ be distinct points on the unit circle with center O. For every two distinct integers i, j, let P_{ij} be the midpoint of A_i and A_j. Find the smallest possible value of

$$\sum_{1 \leq i < j \leq 2015} OP_{ij}^2.$$

23. [**14**] Let $S = \{1, 2, 4, 8, 16, 32, 64, 128, 256\}$. A subset P of S is called *squarely* if it is nonempty and the sum of its elements is a perfect square. A squarely set Q is called *super squarely* if it is not a proper subset of any squarely set. Find the number of super squarely sets.

    (A set A is said to be a *proper subset* of a set B if A is a subset of B and $A \neq B$.)

24. [**14**] $ABCD$ is a cyclic quadrilateral with sides $AB = 10$, $BC = 8$, $CD = 25$, and $DA = 12$. A circle ω is tangent to segments DA, AB, and BC. Find the radius of ω.

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25. [**17**] Let $r_1, \ldots, r_n$ be the distinct real zeroes of the equation

$$x^8 - 14x^4 - 8x^3 - x^2 + 1 = 0.$$

Evaluate $r_1^2 + \cdots + r_n^2$.

26. [**17**] Let $a = \sqrt{17}$ and $b = i\sqrt{19}$, where $i = \sqrt{-1}$. Find the maximum possible value of the ratio $|a - z|/|b - z|$ over all complex numbers z of magnitude 1 (i.e. over the unit circle $|z| = 1$).

27. [**17**] Let a, b be integers chosen independently and uniformly at random from the set $\{0, 1, 2, \ldots, 80\}$. Compute the expected value of the remainder when the binomial coefficient $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ is divided by 3. (Here $\binom{0}{0} = 1$ and $\binom{a}{b} = 0$ whenever $a < b$.)

28. [**17**] Let w, x, y, and z be positive real numbers such that

$$\begin{aligned} 0 &\neq \cos w \cos x \cos y \cos z \\ 2\pi &= w + x + y + z \\ 3 \tan w &= k(1 + \sec w) \\ 4 \tan x &= k(1 + \sec x) \\ 5 \tan y &= k(1 + \sec y) \\ 6 \tan z &= k(1 + \sec z). \end{aligned}$$

(Here $\sec t$ denotes $\frac{1}{\cos t}$ when $\cos t \neq 0$.) Find k.

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29. [**20**] Let ABC be a triangle whose incircle has center I and is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, at D, E, F. Denote by X the midpoint of major arc $\widehat{BAC}$ of the circumcircle of ABC. Suppose P is a point on line XI such that $\overline{DP} \perp \overline{EF}$.

    Given that $AB = 14$, $AC = 15$, and $BC = 13$, compute DP.

30. [**20**] Find the sum of squares of all **distinct** complex numbers x satisfying the equation

$$0 = 4x^{10} - 7x^9 + 5x^8 - 8x^7 + 12x^6 - 12x^5 + 12x^4 - 8x^3 + 5x^2 - 7x + 4.$$

31. [**20**] Define a *power cycle* to be a set S consisting of the nonnegative integer powers of an integer a, i.e. $S = \{1, a, a^2, \dots\}$ for some integer a. What is the minimum number of power cycles required such that given any odd integer n, there exists some integer k in one of the power cycles such that $n \equiv k \pmod{1024}$?

32. [**20**] A wealthy king has his blacksmith fashion him a large cup, whose inside is a cone of height 9 inches and base diameter 6 inches (that is, the opening at the top of the cup is 6 inches in diameter). At one of his many feasts, he orders the mug to be filled to the brim with cranberry juice.

    For each positive integer n, the king stirs his drink vigorously and takes a sip such that the height of fluid left in his cup after the sip goes down by $\frac{1}{n^2}$ inches. Shortly afterwards, while the king is distracted, the court jester adds pure Soylent to the cup until it's once again full. The king takes sips precisely every minute, and his first sip is exactly one minute after the feast begins.

    As time progresses, the amount of juice consumed by the king (in cubic inches) approaches a number r. Find r.

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33. [**25**] Let N denote the sum of the decimal digits of $\binom{1000}{100}$. Estimate the value of N. If your answer is a positive integer A written fully in decimal notation (for example, 521495223), your score will be the greatest integer not exceeding $25 \cdot (0.99)^{|A-N|}$. Otherwise, your score will be zero.

34. [**25**] For an integer n, let $f(n)$ denote the number of pairs (x, y) of integers such that $x^2 + xy + y^2 = n$. Compute the sum

$$\sum_{n=1}^{10^6} nf(n).$$

Write your answer in the form $a \cdot 10^b$, where b is an integer and $1 \le a < 10$ is a decimal number.

If your answer is written in this form, your score will be $\max\{0, 25 - \lfloor 100|\log_{10}(A/N)|\rfloor\}$, where $N = a \cdot 10^b$ is your answer to this problem and A is the actual answer. Otherwise, your score will be zero.

35. [**25**] Let P denote the set of all subsets of $\{1, \ldots, 23\}$. A subset $S \subseteq P$ is called *good* if whenever A, B are sets in S, the set $(A \setminus B) \cup (B \setminus A)$ is also in S. (Here, $A \setminus B$ denotes the set of all elements in A that are not in B, and $B \setminus A$ denotes the set of all elements in B that are not in A.) What **fraction** of the good subsets of P have between 2015 and 3015 elements, inclusive?

If your answer is a decimal number or a fraction (of the form m/n, where m and n are positive integers), then your score on this problem will be equal to $\max\{0, 25 - \lfloor 1000|A - N|\rfloor\}$, where N is your answer and A is the actual answer. Otherwise, your score will be zero.

36. [**25**] A prime number p is *twin* if at least one of $p+2$ or $p-2$ is prime and *sexy* if at least one of $p+6$ and $p-6$ is prime.

How many sexy twin primes (i.e. primes that are both twin and sexy) are there less than 10^9? Express your answer as a positive integer N in decimal notation; for example, 521495223. If your answer is in this form, your score for this problem will be $\max\left\{0, 25 - \lfloor \frac{1}{10000}|A - N|\rfloor\right\}$, where A is the actual answer to this problem. Otherwise, your score will be zero.