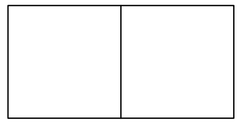
HMMT February 2015

Saturday 21 February 2015

Guts

1. [**4**] Let R be the rectangle in the Cartesian plane with vertices at $(0,0),(2,0),(2,1)$, and $(0,1)$. R can be divided into two unit squares, as shown.



The resulting figure has 7 segments of unit length, connecting neighboring lattice points (those lying on or inside R). Compute the number of paths from $(0,1)$ (the upper left corner) to $(2,0)$ (the lower right corner) along these 7 segments, where each segment can be used at most once.

Answer: $\boxed{4}$ Just count them directly. If the first step is to the right, there are 2 paths. If the first step is downwards (so the next step must be to the right), there are again 2 paths. This gives a total of 4 paths.

2. [**4**] Let $ABCDE$ be a convex pentagon such that $\angle ABC = \angle ACD = \angle ADE = 90^\circ$ and $AB = BC = CD = DE = 1$. Compute AE.

Answer: $\boxed{2}$ By Pythagoras,

$$AE^2 = AD^2 + 1 = AC^2 + 2 = AB^2 + 3 = 4,$$

so $AE = 2$.

3. [**4**] Find the number of pairs of union/intersection operations $(\Box_1, \Box_2) \in \{\cup, \cap\}^2$ satisfying the following condition: for any sets S, T, function $f : S \to T$, and subsets X, Y, Z of S, we have equality of **sets**

$$f(X)\Box_1(f(Y)\Box_2 f(Z)) = f\left(X\Box_1(Y\Box_2 Z)\right),$$

where $f(X)$ denotes the *image* of X: the **set** $\{f(x) : x \in X\}$, which is a subset of T. The images $f(Y)$ (of Y) and $f(Z)$ (of Z) are similarly defined.

Answer: $\boxed{1}$ If and only if $\Box_1 = \Box_2 = \cup$. See `http://math.stackexchange.com/questions/359693/overview-of-`

4. [**4**] Consider the function $z(x,y)$ describing the paraboloid

$$z = (2x - y)^2 - 2y^2 - 3y.$$

Archimedes and Brahmagupta are playing a game. Archimedes first chooses x. Afterwards, Brahmagupta chooses y. Archimedes wishes to minimize z while Brahmagupta wishes to maximize z. Assuming that Brahmagupta will play optimally, what value of x should Archimedes choose?

Answer: $\boxed{-\frac{3}{8}}$ Viewing x as a constant and completing the square, we find that

$$\begin{aligned} z &= 4x^2 - 4xy + y^2 - 2y^2 - 3y \\ &= -y^2 - (4x+3)y + 4x^2 \\ &= -\left(y + \frac{4x+3}{2}\right)^2 + \left(\frac{4x+3}{2}\right)^2 + 4x^2. \end{aligned}$$

Bhramagupta wishes to maximize z, so regardless of the value of x, he will pick $y = -\frac{4x+3}{2}$. The expression for z then simplifies to

$$z = 8x^2 + 6x + \frac{9}{4}.$$

Archimedes knows this and will therefore pick x to minimize the above expression. By completing the square, we find that $x = -\frac{3}{8}$ minimizes z.

Alternatively, note that z is convex in x and concave in y, so we can use the minimax theorem to switch the order of moves. If Archimedes goes second, he will set $x = \frac{y}{2}$ to minimize z, so Brahmagupta will maximize $-2y^2 - 3y$ by setting $y = -\frac{3}{4}$. Thus Archimedes should pick $x = -\frac{3}{8}$, as above.

5. [**5**] Let $\mathcal{H}$ be the unit hypercube of dimension 4 with a vertex at (x, y, z, w) for each choice of $x, y, z, w \in \{0, 1\}$. (Note that $\mathcal{H}$ has $2^4 = 16$ vertices.) A bug starts at the vertex $(0, 0, 0, 0)$. In how many ways can the bug move to $(1, 1, 1, 1)$ (the opposite corner of $\mathcal{H}$) by taking exactly 4 steps along the edges of $\mathcal{H}$?

   Answer: $\boxed{24}$ You may think of this as sequentially adding 1 to each coordinate of $(0, 0, 0, 0)$. There are 4 ways to choose the first coordinate, 3 ways to choose the second, and 2 ways to choose the third. The product is 24.

6. [**5**] Let D be a regular ten-sided polygon with edges of length 1. A triangle T is defined by choosing three vertices of D and connecting them with edges. How many different (non-congruent) triangles T can be formed?

   Answer: $\boxed{8}$ The problem is equivalent to finding the number of ways to partition 10 into a sum of three (unordered) positive integers. These can be computed by hand to be $(1, 1, 8)$, $(1, 2, 7)$, $(1, 3, 6)$, $(1, 4, 5)$, $(2, 2, 6)$, $(2, 3, 5)$, $(2, 4, 4)$, $(3, 3, 4)$.

7. [**5**] Let $\mathcal{C}$ be a cube of side length 2. We color each of the faces of $\mathcal{C}$ blue, then subdivide it into $2^3 = 8$ unit cubes. We then randomly rearrange these cubes (possibly with rotation) to form a new 3-dimensional cube.

   What is the probability that its exterior is still completely blue?

   Answer: $\boxed{\frac{1}{2^{24}} \text{ or } \frac{1}{8^8} \text{ or } \frac{1}{16777216}}$ Each vertex of the original cube must end up as a vertex of the new cube in order for all the old blue faces to show. There are 8 such vertices, each corresponding to one unit cube, and each has a probability $\frac{1}{8}$ of being oriented with the old outer vertex as a vertex of the new length-2 cube. Multiplying gives the answer.

8. [**5**] Evaluate
$$\sin(\arcsin(0.4) + \arcsin(0.5)) \cdot \sin(\arcsin(0.5) - \arcsin(0.4)),$$
   where for $x \in [-1, 1]$, $\arcsin(x)$ denotes the unique real number $y \in [-\pi, \pi]$ such that $\sin(y) = x$.

   Answer: $\boxed{0.09 \text{ OR } \frac{9}{100}}$ Use the difference of squares identity[1] $\sin(a-b)\sin(a+b) = \sin(a)^2 - \sin(b)^2$ to get $0.5^2 - 0.4^2 = 0.3^2 = 0.09 = \frac{9}{100}$.

9. [**6**] Let a, b, c be integers. Define $f(x) = ax^2 + bx + c$. Suppose there exist pairwise distinct integers u, v, w such that $f(u) = 0$, $f(v) = 0$, and $f(w) = 2$. Find the maximum possible value of the discriminant $b^2 - 4ac$ of f.

   Answer: $\boxed{16}$ By the factor theorem, $f(x) = a(x - u)(x - v)$, so the constraints essentially boil down to $2 = f(w) = a(w - u)(w - v)$. (It's not so important that $u \neq v$; we merely specified it for a shorter problem statement.)

   We want to maximize the discriminant $b^2 - 4ac = a^2[(u+v)^2 - 4uv] = a^2(u-v)^2 = a^2[(w-v)-(w-u)]^2$. Clearly $a \mid 2$. If $a > 0$, then $(w - u)(w - v) = 2/a > 0$ means the difference $|u - v|$ is less than $2/a$, whereas if $a < 0$, since at least one of $|w - u|$ and $|w - v|$ equals 1, the difference $|u - v|$ of factors is greater than $2/|a|$.

   So the optimal choice occurs either for $a = -1$ and $|u - v| = 3$, or $a = -2$ and $|u - v| = 2$. The latter wins, giving a discriminant of $(-2)^2 \cdot 2^2 = 16$.

[1]proven most easily with complex numbers or the product-to-sum identity on $\sin(a - b)\sin(a + b)$ (followed by the double angle formula for cosine)

10. [**6**] Let $b(x) = x^2 + x + 1$. The polynomial $x^{2015} + x^{2014} + \cdots + x + 1$ has a unique "base $b(x)$" representation

$$x^{2015} + x^{2014} + \cdots + x + 1 = \sum_{k=0}^{N} a_k(x) b(x)^k,$$

where

- N is a nonnegative integer;
- each "digit" $a_k(x)$ (for $0 \le k \le N$) is either the zero polynomial (i.e. $a_k(x) = 0$) or a nonzero polynomial of degree less than $\deg b = 2$; and
- the "leading digit $a_N(x)$" is nonzero (i.e. not the zero polynomial).

Find $a_N(0)$ (the "leading digit evaluated at 0").

Answer: $\boxed{-1006}$ Comparing degrees easily gives $N = 1007$. By ignoring terms of degree at most 2013, we see

$$a_N(x)(x^2 + x + 1)^{1007} \in x^{2015} + x^{2014} + O(x^{2013}).$$

Write $a_N(x) = ux + v$, so

$$\begin{aligned} a_N(x)(x^2 + x + 1)^{1007} &\in (ux + v)(x^{2014} + 1007x^{2013} + O(x^{2012})) \\ &\subseteq ux^{2015} + (v + 1007u)x^{2014} + O(x^{2013}). \end{aligned}$$

Finally, matching terms gives $u = 1$ and $v + 1007u = 1$, so $v = 1 - 1007 = -1006$.

Remark. This problem illustrates the analogy between integers and polynomials, with the nonconstant (degree ≥ 1) polynomial $b(x) = x^2 + x + 1$ taking the role of a positive integer base $b > 1$.

11. [**6**] Find

$$\sum_{k=0}^{\infty} \left\lfloor \frac{1 + \sqrt{\frac{2000000}{4^k}}}{2} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

Answer: $\boxed{1414}$ The kth floor (for $k \ge 0$) counts the number of **positive integer** solutions to $4^k(2x-1)^2 \le 2 \cdot 10^6$. So summing over all k, we want the number of integer solutions to $4^k(2x-1)^2 \le 2 \cdot 10^6$ with $k \ge 0$ and $x \ge 1$. But each positive integer can be uniquely represented as a power of 2 times an odd (positive) integer, so there are simply $\lfloor 10^3\sqrt{2} \rfloor = 1414$ solutions.

12. [**6**] For integers a, b, c, d, let $f(a, b, c, d)$ denote the number of ordered pairs of integers $(x, y) \in \{1, 2, 3, 4, 5\}^2$ such that $ax + by$ and $cx + dy$ are both divisible by 5. Find the sum of all possible values of $f(a, b, c, d)$.

Answer: $\boxed{31}$ Standard linear algebra over the field $\mathbb{F}_5$ (the integers modulo 5). The dimension of the solution set is at least 0 and at most 2, and any intermediate value can also be attained. So the answer is $1 + 5 + 5^2 = 31$.

This also can be easily reformulated in more concrete equation/congruence-solving terms, especially since there are few variables/equations.

13. [**8**] Let $P(x) = x^3 + ax^2 + bx + 2015$ be a polynomial all of whose roots are integers. Given that $P(x) \ge 0$ for all $x \ge 0$, find the sum of all possible values of $P(-1)$.

Answer: $\boxed{9496}$ Since all the roots of $P(x)$ are integers, we can factor it as $P(x) = (x-r)(x-s)(x-t)$ for integers r, s, t. By Viete's formula, the product of the roots is $rst = -2015$, so we need three integers to multiply to -2015.

$P(x)$ cannot have two distinct positive roots u, v since otherwise, $P(x)$ would be negative at least in some infinitesimal region $x < u$ or $x > v$, or $P(x) < 0$ for $u < x < v$. Thus, in order to have two positive roots, we must have a double root. Since $2015 = 5 \times 13 \times 31$, the only positive double root is a perfect square factor of 2015, which is at $x = 1$, giving us a possibility of $P(x) = (x-1)^2(x+2015)$.

Now we can consider when $P(x)$ only has negative roots. The possible unordered triplets are $(-1,-1,-2015), (-1,-5,-4$ $(-1,-31,-65), (-5,-13,-31)$, which yield the polynomials $(x+1)^2(x+2015), (x+1)(x+5)(x+403), (x+1)(x+13)(x+155), (x+1)(x+31)(x+65), (x+5)(x+13)(x+31)$, respectively.

Noticing that $P(-1) = 0$ for four of these polynomials, we see that the nonzero values are $P(-1) = (-1-1)^2(2014), (5-1)(13-1)(31-1)$, which sum to $8056 + 1440 = 9496$.

14. [**8**] Find the smallest integer $n \geq 5$ for which there exists a set of n distinct pairs $(x_1, y_1), \ldots, (x_n, y_n)$ of positive integers with $1 \leq x_i, y_i \leq 4$ for $i = 1, 2, \ldots, n$, such that for any indices $r, s \in \{1, 2, \ldots, n\}$ (not necessarily distinct), there exists an index $t \in \{1, 2, \ldots, n\}$ such that 4 divides $x_r + x_s - x_t$ and $y_r + y_s - y_t$.

Answer: $\boxed{8}$ In other words, we have a set S of n pairs in $(\mathbb{Z}/4\mathbb{Z})^2$ closed under addition. Since $1+1+1+1 \equiv 0 \pmod 4$ and $1+1+1 \equiv -1 \pmod 4$, $(0,0) \in S$ and S is closed under (additive) inverses. Thus S forms a group under addition (a subgroup of $(\mathbb{Z}/4\mathbb{Z})^2$). By Lagrange's theorem (from basic group theory), $n \mid 4^2$, so $n \geq 8$. To achieve this bound, one possible construction is $\{1,2,3,4\} \times \{2,4\}$.

Remark. In fact, S is a finite abelian group. Such groups have a very clean classification; this is clarified by the fact that abelian groups are the same as modules over $\mathbb{Z}$, the ring of integers.

15. [**8**] Find the maximum possible value of $H \cdot M \cdot M \cdot T$ over all ordered triples (H, M, T) of integers such that $H \cdot M \cdot M \cdot T = H + M + M + T$.

Answer: $\boxed{8}$ If any of H, M, T are zero, the product is 0. We can do better (examples below), so we may now restrict attention to the case when $H, M, T \neq 0$.

When $M \in \{-2,-1,1,2\}$, a little casework gives all the possible $(H,M,T) = (2,1,4), (4,1,2), (-1,-2,1), (1,-2,-1)$.

- If $M = -2$, i.e. $H - 4 + T = 4HT$, then $-15 = (4H-1)(4T-1)$, so $4H - 1 \in \{\pm 1, \pm 3, \pm 5, \pm 15\}$ (only $-1, +3, -5, +15$ are possible) corresponding to $4T - 1 \in \{\mp 15, \mp 5, \mp 3, \mp 1\}$ (only $+15, -5, +3, -1$ are possible). But H, T are nonzero, we can only have $4H - 1 \in \{+3, -5\}$, yielding $(-1,-2,1)$ and $(1,-2,-1)$.
- If $M = +2$, i.e. $H + 4 + T = 4HT$, then $17 = (4H-1)(4T-1)$, so $4H - 1 \in \{\pm 1, \pm 17\}$ (only $-1, -17$ are possible) corresponding to $4T - 1 \in \{\pm 17, \pm 1\}$ (only $-17, -1$ are possible). But H, T are nonzero, so there are no possibilities here.
- If $M = -1$, i.e. $H - 2 + T = HT$, then $-1 = (H-1)(T-1)$, so we have $H - 1 \in \{\pm 1\}$ and $T - 1 \in \{\mp 1\}$, neither of which is possible (as $H, T \neq 0$).
- If $M = +1$, i.e. $H + 2 + T = HT$, then $3 = (H-1)(T-1)$, so we have $H - 1 \in \{\pm 1, \pm 3\}$. Since $H, T \neq 0$, $H - 1 \in \{+1, +3\}$, yielding $(2,1,4)$ and $(4,1,2)$.

Now suppose there is such a triple (H, M, T) for $|M| \geq 3$. The equation in the problem gives $(M^2H - 1)(M^2T - 1) = 2M^3 + 1$. Note that since $H, T \neq 0$, $|2M^3 + 1| = |M^2H - 1| \cdot |M^2T - 1| \geq \min(M^2 - 1, M^2 + 1)^2 = M^4 - 2M^2 + 1 > 2|M|^3 + 1$ gives a contradiction.

16. [**8**] Determine the number of **unordered** triples of **distinct** points in the $4 \times 4 \times 4$ lattice grid $\{0,1,2,3\}^3$ that are collinear in $\mathbb{R}^3$ (i.e. there exists a line passing through the three points).

Answer: $\boxed{376}$ Define a main plane to be one of the xy, yz, zx planes. Define a space diagonal to be a set of collinear points not parallel to a main plane. We classify the lines as follows:

(a) **Lines parallel to two axes (i.e. orthogonal to a main plane).** Notice that given a plane of the form $v = k$, where $v \in \{x,y,z\}$, $k \in \{0,1,2,3\}$, there are 8 such lines, four in one direction and four in a perpendicular direction. There are $4 \times 3 = 12$ such planes. However, each line lies in two of these (v,k) planes, so there are $\frac{8 \times 4 \times 3}{2} = 48$ such lines. Each of these lines has 4 points, so there are 4 possible ways to choose 3 collinear points, giving $4 \times 48 = 192$ triplets.

(b) **Diagonal lines containing four points parallel to some main plane.** Consider a plane of the form (v, k), as defined above. These each have 2 diagonals that contain 4 collinear points. Each of these diagonals uniquely determines v, k so these diagonals are each counted once. There are 12 possible (v, k) pairs, yielding $12 \times 2 \times 4 = 96$ triplets.

(c) **Diagonal lines containing three points parallel to some main plane.** Again, consider a plane (v, k). By inspection, there are four such lines and one way to choose the triplet of points for each of these lines. This yields $4 \times 12 = 48$ triplets.

(d) **Main diagonals.** There are four main diagonals, each with 4 collinear points, yielding $4 \times 4 = 16$ triplets.

(e) **Space diagonals containing three points.** Choose one of the points in the set $\{1, 2\}^3$ to be the midpoint of the line. Since these 8 possibilities are symmetric, say we take the point $(1, 1, 1)$. There are four space diagonals passing through this point, but one is a main diagonal. So each of the 8 points has 3 such diagonals with 3 points each, yielding $8 \times 3 = 24$ ways.

Adding all these yields $192 + 96 + 48 + 16 + 24 = 376$.

17. [**11**] Find the least positive integer $N > 1$ satisfying the following two properties:

- There exists a positive integer a such that $N = a(2a - 1)$.
- The sum $1 + 2 + \cdots + (N - 1)$ is divisible by k for every integer $1 \leq k \leq 10$.

Answer: $\boxed{2016}$ The second condition implies that 16 divides $a(2a-1)(2a^2 - a - 1)$, which shows that $a \equiv 0$ or 1 modulo 16. The case $a = 1$ would contradict the triviality-avoiding condition $N > 1$. a cannot be 16, because 7 does not divide $a(2a-1)(2a^2 - a - 1)$. a cannot be 17, because 9 does not divide $a(2a-1)(2a^2 - a - 1)$. It can be directly verified that $a = 32$ is the smallest positive integer for which $1 + 2 + \cdots + (N - 1) = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 31$ which is divisible by $1, 2, \ldots, 10$. For this a, we compute $N = 32(2 \cdot 32 - 1) = 2016$.

18. [**11**] Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function such that for any integers x, y, we have

$$f(x^2 - 3y^2) + f(x^2 + y^2) = 2(x + y)f(x - y).$$

Suppose that $f(n) > 0$ for all $n > 0$ and that $f(2015) \cdot f(2016)$ is a perfect square. Find the minimum possible value of $f(1) + f(2)$.

Answer: $\boxed{246}$ Plugging in $-y$ in place of y in the equation and comparing the result with the original equation gives

$$(x - y)f(x + y) = (x + y)f(x - y)$$

This shows that whenever $a, b \in \mathbb{Z} - \{0\}$ with $a \equiv b (\text{mod } 2)$, we have

$$\frac{f(a)}{a} = \frac{f(b)}{b}$$

which implies that there are constants $\alpha = f(1) \in \mathbb{Z}_{>0}, \beta = f(2) \in \mathbb{Z}_{>0}$ for which f satisfies the equation $(*)$:

$$f(n) = \begin{cases} n \cdot \alpha & \text{when } 2 \nmid n \\ \frac{n}{2} \cdot \beta & \text{when } 2 | n \end{cases}$$

Therefore, $f(2015)f(2016) = 2015\alpha \cdot 1008\beta = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 31\alpha\beta$, so $\alpha\beta = 5 \cdot 7 \cdot 13 \cdot 31 \cdot t^2$ for some $t \in \mathbb{Z}_{>0}$. We claim that $(\alpha, \beta, t) = (5 \cdot 31, 7 \cdot 13, 1)$ is a triple which gives the minimum $\alpha + \beta$. In particular, we claim $\alpha + \beta \geq 246$.

Consider the case $t \geq 2$ first. We have, by AM-GM, $\alpha + \beta \geq 2 \cdot \sqrt{\alpha\beta} \geq 4 \cdot \sqrt{14105} > 246$. Suppose $t = 1$. We have $\alpha \cdot \beta = 5 \cdot 7 \cdot 13 \cdot 31$. Because $(\alpha + \beta)^2 - (\alpha - \beta)^2 = 4\alpha\beta$ is fixed, we want to have α as close as β as possible. This happens when one of α, β is $5 \cdot 31$ and the other is $7 \cdot 13$. In this case, $\alpha + \beta = 91 + 155 = 246$.

Finally, we note that the equality $f(1)+f(2)=246$ can be attained. Consider $f:\mathbb{Z}\to\mathbb{Z}$ such that $f(n)=91n$ for every odd $n\in\mathbb{Z}$ and $f(n)=\frac{155}{2}n$ for every even $n\in\mathbb{Z}$. It can be verified that f satisfies the condition in the problem and $f(1)+f(2)=246$ as claimed.

19. [**11**] Find the smallest positive integer n such that the polynomial $(x+1)^n-1$ is "divisible by x^2+1 modulo 3", or more precisely, either of the following **equivalent** conditions holds:

- there exist polynomials P,Q with integer coefficients such that $(x+1)^n-1=(x^2+1)P(x)+3Q(x)$;
- or more conceptually, the remainder when (the polynomial) $(x+1)^n-1$ is divided by (the polynomial) x^2+1 is a polynomial with (integer) coefficients all divisible by 3.

Answer: $\boxed{8}$ **Solution 1.** We have $(x+1)^2=x^2+2x+1\equiv 2x$, $(x+1)^4\equiv(2x)^2\equiv-4\equiv-1$, and $(x+1)^8\equiv(-1)^2=1$. So the order n divides 8, as $x+1$ and x^2+1 are relatively prime polynomials modulo 3 (or more conceptually, in $\mathbb{F}_3[x]$), but cannot be smaller by our computations of the 2nd and 4th powers.

Remark. This problem illustrates the analogy between integers and polynomials (specifically here, polynomials over the *finite field* of integers modulo 3), with $x^2+1 \pmod 3$ taking the role of a prime number. Indeed, we can prove analogously to Fermat's little theorem that $f(x)^{3^{\deg(x^2+1)}}\equiv f(x) \pmod{x^2+1,3}$ for any polynomial $f(x)$: try to work out the proof yourself! (This kind of problem, with 3 replaced by any prime p and x^2+1 by any irreducible polynomial modulo p, is closely related to the theory of *(extensions of) finite fields.*)

Solution 2. Here's a solution avoiding the terminology of groups and fields. Let $R(x)=(x+1)^n-1-P(x)(x^2+1)=3Q(x)$ be the remainder (here P is the quotient), whose coefficients (upon expansion) must all be divisible by 3. Now consider $R(i)$. The coefficients of the even and odd powers of x are divisible by 3, so $R(i)$ must have real and imaginary parts divisible by 3. Notice that $(1+i)^2=2i$, $(1+i)^4=-4$, $(1+i)^6=-8i$, $(1+i)^8=16$, and that $(1+i)^8-1=15$, which has real and imaginary parts divisible by 3. Since any even power of $(1+i)$ (for $n\le 6$) yields a purely real or purely imaginary number with a coefficient not divisible by 3, multiplying it by $1+i$ will yield an imaginary part not divisible by 3. To see that $n=8$ works, notice that, when taken modulo 3,

$$\begin{aligned}(x+1)^8-1&=x^8+8x^7+28x^6+56x^5+70x^4+56x^3+28x^2+8x\\&\equiv x^8-x^7+x^6-x^5+x^4-x^3+x^2-x\\&=(x^2+1)(x^6-x^5+x^2-x)\pmod 3,\end{aligned}$$

as desired.

20. [**11**] What is the largest real number θ less than π (i.e. $\theta<\pi$) such that

$$\prod_{k=0}^{10}\cos(2^k\theta)\neq 0$$

and

$$\prod_{k=0}^{10}\left(1+\frac{1}{\cos(2^k\theta)}\right)=1?$$

Answer: $\boxed{\frac{2046\pi}{2047}}$ For equality to hold, note that θ cannot be an integer multiple of π (or else $\sin=0$ and $\cos=\pm1$).

Let $z=e^{i\theta/2}\neq\pm1$. Then in terms of complex numbers, we want

$$\prod_{k=0}^{10}\left(1+\frac{2}{z^{2^{k+1}}+z^{-2^{k+1}}}\right)=\prod_{k=0}^{10}\frac{(z^{2^k}+z^{-2^k})^2}{z^{2^{k+1}}+z^{-2^{k+1}}},$$

which partially telescopes to

$$\frac{z+z^{-1}}{z^{2^{11}}+z^{-2^{11}}}\prod_{k=0}^{10}(z^{2^k}+z^{-2^k}).$$

Using a classical telescoping argument (or looking at binary representation; if you wish we may note that $z-z^{-1}\neq 0$, so the ultimate telescoping identity holds[2]), this simplifies to

$$\frac{z+z^{-1}}{z^{2^{11}}+z^{-2^{11}}}\frac{z^{2^{11}}-z^{-2^{11}}}{z-z^{-1}}=\frac{\tan(2^{10}\theta)}{\tan(\theta/2)}.$$

Since $\tan x$ is injective modulo π (i.e. π-periodic and injective on any given period), θ works if and only if $\frac{\theta}{2}+\ell\pi=1024\theta$ for some integer ℓ, so $\theta=\frac{2\ell\pi}{2047}$. The largest value for ℓ such that $\theta<\pi$ is at $\ell=1023$, which gives $\theta=\frac{2046\pi}{2047}$.[3]

Remark. It's also possible to do this without complex numbers, but it's less systematic. The steps are the same, though, first note that $1+\sec 2^k\theta=\frac{1+\cos 2^k\theta}{\cos 2^k\theta}=\frac{2\cos^2 2^{k-1}\theta}{\cos 2^k\theta}$ using the identity $\cos 2x=2\cos^2 x-1$ (what does this correspond to in complex numbers?). hen we telescope using the identity $2\cos x=\frac{\sin 2x}{\sin x}$ (again, what does this correspond to in complex numbers?).

21. [**14**] Define a sequence $a_{i,j}$ of integers such that $a_{1,n}=n^n$ for $n\geq 1$ and $a_{i,j}=a_{i-1,j}+a_{i-1,j+1}$ for all $i,j\geq 1$. Find the last (decimal) digit of $a_{128,1}$.

Answer: $\boxed{4}$ By applying the recursion multiple times, we find that $a_{1,1}=1$, $a_{2,n}=n^n+(n+1)^{n+1}$, and $a_{3,n}=n^n+2(n+1)^{n+1}+(n+2)^{n+2}$. At this point, we can conjecture and prove by induction that

$$a_{m,n}=\sum_{k=0}^{m-1}\binom{m-1}{k}(n+k)^{n+k}=\sum_{k\geq 0}\binom{m-1}{k}(n+k)^{n+k}.$$

(The second expression is convenient for dealing with boundary cases. The induction relies on $\binom{m}{0}=\binom{m-1}{0}$ on the $k=0$ boundary, as well as $\binom{m}{k}=\binom{m-1}{k}+\binom{m-1}{k-1}$ for $k\geq 1$.) We fix $m=128$. Note that $\binom{127}{k}\equiv 1 \pmod 2$ for all $1\leq k\leq 127$ and $\binom{127}{k}\equiv 0 \pmod 5$ for $3\leq k\leq 124$, by Lucas' theorem on binomial coefficients. Therefore, we find that

$$a_{128,1}=\sum_{k=0}^{127}\binom{127}{k}(k+1)^{k+1}\equiv\sum_{k=0}^{127}(k+1)^{k+1}\equiv 0 \pmod 2$$

and

$$a_{128,1}\equiv\sum_{k\in[0,2]\cup[125,127]}\binom{127}{k}(k+1)^{k+1}\equiv 4 \pmod 5.$$

Therefore, $a_{128,1}\equiv 4 \pmod{10}$.

22. [**14**] Let $A_1,A_2,\ldots,A_{2015}$ be distinct points on the unit circle with center O. For every two distinct integers i,j, let P_{ij} be the midpoint of A_i and A_j. Find the smallest possible value of

$$\sum_{1\leq i<j\leq 2015}OP_{ij}^2.$$

Answer: $\boxed{\frac{2015\cdot 2013}{4}\text{ OR }\frac{4056195}{4}}$ Use vectors. $\sum|a_i+a_j|^2/4=\sum(2+2a_i\cdot a_j)/4=\frac{1}{2}\binom{2015}{2}+\frac{1}{4}(|\sum a_i|^2-\sum|a_i|^2)\geq 2015\cdot\frac{2014}{4}-\frac{2015}{4}=\frac{2015\cdot 2013}{4}$, with equality if and only if $\sum a_i=0$, which occurs for instance for a regular 2015-gon.

[2] The identity still holds even if $z^{2^k}-z^{-2^k}=0$ for some $k\geq 1$ used in the telescoping argument: why?

[3] This indeed works, since $\prod_{k=0}^{10}\cos(2^k\theta)\neq 0$: why?

23. [**14**] Let $S = \{1, 2, 4, 8, 16, 32, 64, 128, 256\}$. A subset P of S is called *squarely* if it is nonempty and the sum of its elements is a perfect square. A squarely set Q is called *super squarely* if it is not a proper subset of any squarely set. Find the number of super squarely sets.

(A set A is said to be a *proper subset* of a set B if A is a subset of B and $A \neq B$.)

Answer: $\boxed{5}$ Clearly we may biject squarely sets with binary representations of perfect squares between 1 and $2^0 + \cdots + 2^8 = 2^9 - 1 = 511$, so there are 22 squarely sets, corresponding to n^2 for $n = 1, 2, \ldots, 22$. For convenience, we say N is (super) squarely if and only if the set corresponding to N is (super) squarely.

The general strategy is to rule out lots of squares at time, by searching for squares with few missing digits (and ideally most 1's consecutive, for simplicity). We can restrict ourselves (for now) to odds; $(2k)^2$ is just k^2 with two additional zeros at the end. $1, 9, 25, 49, 81$ are ineffective, but $121 = 2^7 - 7 = 2^6 + 2^5 + 2^4 + 2^3 + 2^0$ immediately rules out all odd squares up to 9^2, as they must be $1 \pmod 8$.

Fortunately, $22^2 = 4 \cdot 11^2$ is in our range (i.e. less than 512), ruling out all even squares up to 20^2 as well.

This leaves us with $11^2, 13^2, 15^2, 17^2, 19^2, 21^2, 22^2$, with binary representations 001111001, 010101001, 011100001, 100100001, 101101001 (kills 17^2), 110111001 (kills 13^2), 111100100 (kills nothing by parity). Thus $11^2, 15^2, 19^2, 21^2, 22^2$ are the only super squarely numbers, for a total of 5.

24. [**14**] $ABCD$ is a cyclic quadrilateral with sides $AB = 10$, $BC = 8$, $CD = 25$, and $DA = 12$. A circle ω is tangent to segments DA, AB, and BC. Find the radius of ω.

Answer: $\boxed{\sqrt{\frac{1209}{7}} \text{ OR } \frac{\sqrt{8463}}{7}}$ Denote E an intersection point of AD and BC. Let $x = EA$ and $y = EB$. Because $ABCD$ is a cyclic quadrilateral, $\triangle EAB$ is similar to $\triangle ECD$. Therefore, $\frac{y+8}{x} = \frac{25}{10}$ and $\frac{x+12}{y} = \frac{25}{10}$. We get $x = \frac{128}{21}$ and $y = \frac{152}{21}$. Note that ω is the E-excircle of $\triangle EAB$, so we may finish by standard calculations.

Indeed, first we compute the semiperimeter $s = \frac{EA+AB+BE}{2} = \frac{x+y+10}{2} = \frac{35}{3}$. Now the radius of ω is (by Heron's formula for area)

$$r_E = \frac{[EAB]}{s - AB} = \sqrt{\frac{s(s-x)(s-y)}{s-10}} = \sqrt{\frac{1209}{7}} = \frac{\sqrt{8463}}{7}.$$

25. [**17**] Let $r_1, \ldots, r_n$ be the distinct real zeroes of the equation

$$x^8 - 14x^4 - 8x^3 - x^2 + 1 = 0.$$

Evaluate $r_1^2 + \cdots + r_n^2$.

Answer: $\boxed{8}$ Observe that

$$\begin{aligned} x^8 - 14x^4 - 8x^3 - x^2 + 1 &= (x^8 + 2x^4 + 1) - (16x^4 + 8x^3 + x^2) \\ &= (x^4 + 4x^2 + x + 1)(x^4 - 4x^2 - x + 1). \end{aligned}$$

The polynomial $x^4 + 4x^2 + x + 1 = x^4 + \frac{15}{4}x^2 + (\frac{x}{2} + 1)^2$ has no real roots. On the other hand, let $P(x) = x^4 - 4x^2 - x + 1$. Observe that $P(-\infty) = +\infty > 0$, $P(-1) = -1 < 0$, $P(0) = 1 > 0$, $P(1) = -3 < 0$, $P(+\infty) = +\infty > 0$, so by the intermediate value theorem, $P(x) = 0$ has four distinct real roots, which are precisely the real roots of the original degree 8 equation. By Vieta's formula on $P(x)$,

$$\begin{aligned} r_1^2 + r_2^2 + r_3^2 + r_4^2 &= (r_1 + r_2 + r_3 + r_4)^2 - 2 \cdot (\sum_{i<j} r_i r_j) \\ &= 0^2 - 2(-4) = 8. \end{aligned}$$

26. [**17**] Let $a=\sqrt{17}$ and $b=i\sqrt{19}$, where $i=\sqrt{-1}$. Find the maximum possible value of the ratio $|a-z|/|b-z|$ over all complex numbers z of magnitude 1 (i.e. over the unit circle $|z|=1$).

Answer: $\boxed{\frac{4}{3}}$ Let $|a-z|/|b-z|=k$. We wish to determine the minimum and maximum value of k. Squaring and expansion give:

$$\begin{aligned}|a-z|^2&=|b-z|^2\cdot k^2\\|a|^2-2a\cdot z+1&=(|b|^2-2b\cdot z+1)k^2\\|a|^2+1-(|b|^2+1)k^2&=2(a-bk^2)\cdot z,\end{aligned}$$

where $\cdot$ is a dot product of complex numbers, i.e., the dot product of vectors corresponding to the complex numbers in the complex plane. Now, since z has modulus 1 but can assume any direction, the only constraint on the value of k is

$$||a|^2+1-(|b|^2+1)k^2|\le 2|a-bk^2|.$$

Squaring again and completing the square, the inequality reduces to:

$$\begin{aligned}(|a|^2-1)^2+(|b|^2-1)^2k^4+2(4a\cdot b-(|a|^2+1)(|b|^2+1))k^2&\le 0\\((|a|^2-1)-(|b|^2-1)k^2)^2-4|a-b|^2k^2&\le 0\\|(|a|^2-1)-(|b|^2-1)k^2|&\le 2|a-b|k.\end{aligned}$$

At this stage all the relevant expressions are constant real numbers. Denote, for simplicity, $A=|a|^2-1, B=|b|^2-1$, and $C=|a-b|$. Then, we are looking for k such that $|A-Bk^2|\le 2Ck$. If $B=0$, then $k\ge|\frac{A}{2C}|$, so the minimum value is $|\frac{A}{2C}|$ and the maximum value is $+\infty$. Otherwise, consider

$$\begin{aligned}C^2+AB&=(|a|^2-2a\cdot b+|b|^2)+(|a|^2-1)(|b|^2-1)\\&=|ab|^2-2a\cdot b+1\\&=|\overline{a}b|^2-2\Re(\overline{a}b)+1\\&=|\overline{a}b-1|^2\end{aligned}$$

So let $D=|\overline{a}b-1|=\sqrt{C^2+AB}$. We may assume $B>0$ (the another case is analogous: just substitute A,B with $-A,-B$). Then, k is determined by the following inequalities:

$$\begin{aligned}Bk^2+2Ck-A&\ge 0\\Bk^2-2Ck-A&\le 0\end{aligned}$$

The first inequality gives $k\le\frac{-C-D}{B}$ or $k\ge\frac{-C+D}{B}$, and the second gives $\frac{C-D}{B}\le k\le\frac{C+D}{B}$. Combining, this gives $|\frac{C-D}{B}|\le k\le|\frac{C+D}{B}|$, as claimed.

To summarize the general answer, let $A=|a|^2-1, B=|b|^2-1, C=|a-b|, D=|\overline{a}b-1|$. Then, if $|b|=1$, min is $|\frac{A}{2C}|$ and max is $+\infty$; otherwise, min is $|\frac{C-D}{B}|$ and max is $|\frac{C+D}{B}|$.

In the special case $a=\sqrt{17}$ and $b=\sqrt{19}i$, we have $A=16$, $B=18$, $C=\left|\sqrt{17}-\sqrt{19}i\right|=\sqrt{36}=6$, and $D=\sqrt{17\cdot 19+1}=18$. Thus the answer is $\frac{C+D}{B}=\frac{6+18}{18}=\frac{4}{3}$.

27. [**17**] Let a,b be integers chosen independently and uniformly at random from the set $\{0,1,2,\ldots,80\}$. Compute the expected value of the remainder when the binomial coefficient $\binom{a}{b}=\frac{a!}{b!(a-b)!}$ is divided by 3. (Here $\binom{0}{0}=1$ and $\binom{a}{b}=0$ whenever $a<b$.)

Answer: $\boxed{\frac{1816}{6561}}$ By Lucas' Theorem we're looking at

$$\prod_{i=1}^{4}\binom{a_i}{b_i}$$

where the a_i and b_i are the digits of a and b in base 3. If any $a_i < b_i$, then the product is zero modulo 3.

Otherwise, the potential residues are $\binom{2}{0} = 1$, $\binom{2}{1} = 2$, $\binom{2}{2} = 1$, $\binom{1}{0} = 1$, $\binom{1}{1} = 1$, $\binom{0}{0} = 1$.

So each term in the product has a $\frac{1}{3}$ chance of being zero; given that everything is nonzero, each term has a $\frac{1}{6}$ chance of being 2 and a $\frac{5}{6}$ chance of being 1. The probability that an even number of terms are 1 given that none are zero is then given by the roots of unity filter

$$\frac{(\frac{5}{6} + \frac{1}{6} \cdot (1))^4 + (\frac{5}{6} + \frac{1}{6} \cdot (-1))^4}{2} = \frac{81 + 16}{162} = \frac{97}{162}.$$

Thus the expected value is

$$\left(\frac{2}{3}\right)^4 \left(2 - \frac{97}{162}\right) = \frac{1816}{6561}.$$

28. [**17**] Let w, x, y, and z be positive real numbers such that

$$\begin{aligned} 0 &\neq \cos w \cos x \cos y \cos z \\ 2\pi &= w + x + y + z \\ 3 \tan w &= k(1 + \sec w) \\ 4 \tan x &= k(1 + \sec x) \\ 5 \tan y &= k(1 + \sec y) \\ 6 \tan z &= k(1 + \sec z). \end{aligned}$$

(Here $\sec t$ denotes $\frac{1}{\cos t}$ when $\cos t \neq 0$.) Find k.

Answer: $\boxed{\sqrt{19}}$ From the identity $\tan\frac{u}{2} = \frac{\sin u}{1+\cos u}$, the conditions work out to $3\tan\frac{w}{2} = 4\tan\frac{x}{2} = 5\tan\frac{y}{2} = 6\tan\frac{z}{2} = k$. Let $a = \tan\frac{w}{2}$, $b = \tan\frac{x}{2}$, $c = \tan\frac{y}{2}$, and $d = \tan\frac{z}{2}$. Using the identity $\tan(M+N) = \frac{\tan M + \tan N}{1 - \tan M \tan N}$, we obtain

$$\begin{aligned} \tan\left(\frac{w+x}{2} + \frac{y+z}{2}\right) &= \frac{\tan\left(\frac{w+x}{2}\right) + \tan\left(\frac{y+z}{2}\right)}{1 - \tan\left(\frac{w+x}{2}\right)\tan\left(\frac{y+z}{2}\right)} \\ &= \frac{\frac{a+b}{1-ab} + \frac{c+d}{1-cd}}{1 - \left(\frac{a+b}{1-ab}\right)\left(\frac{c+d}{1-cd}\right)} \\ &= \frac{a+b+c+d-abc-abd-bcd-acd}{1+abcd-ab-ac-ad-bc-bd-cd}. \end{aligned}$$

Because $x + y + z + w = \pi$, we get that $\tan\left(\frac{x+y+z+w}{2}\right) = 0$ and thus $a+b+c+d = abc+abd+bcd+acd$. Substituting a, b, c, d corresponding to the variable k, we obtain that $k^3 - 19k = 0$. Therefore, k can be only $0, \sqrt{19}, -\sqrt{19}$. However, $k = 0$ is impossible as w, x, y, z will all be 0. Also, $k = -\sqrt{19}$ is impossible as w, x, y, z will exceed π. Therefore, $k = \sqrt{19}$.

29. [**20**] Let ABC be a triangle whose incircle has center I and is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, at D, E, F. Denote by X the midpoint of major arc $\widehat{BAC}$ of the circumcircle of ABC. Suppose P is a point on line XI such that $\overline{DP} \perp \overline{EF}$.

Given that $AB = 14$, $AC = 15$, and $BC = 13$, compute DP.

Answer: $\boxed{\frac{4\sqrt{5}}{5}}$ Let H be the orthocenter of triangle DEF. We claim that P is the midpoint of $\overline{DH}$. Indeed, consider an inversion at the incicrle of ABC, denoting the inverse of a point with an asterik. It maps ABC to the nine-point circle of $\triangle DEF$. According to $\angle IAX = 90^\circ$, we have $\angle A^*X^*I = 90^\circ$. Hence line XI passes through the point diametrically opposite to A^*, which is the midpoint of $\overline{DH}$, as claimed.

The rest is a straightforward computation. The inradius of $\triangle ABC$ is $r = 4$. The length of EF is given by $EF = 2\frac{AF \cdot r}{AI} = \frac{16}{\sqrt{5}}$. Then,

$$DP^2 = \left(\frac{1}{2}DH\right)^2 = \frac{1}{4}\left(4r^2 - EF^2\right) = 4^2 - \frac{64}{5} = \frac{16}{5}.$$

Hence $DP = \frac{4\sqrt{5}}{5}$.

Remark. This is also not too bad of a coordinate bash.

30. [**20**] Find the sum of squares of all **distinct** complex numbers x satisfying the equation

$$0 = 4x^{10} - 7x^9 + 5x^8 - 8x^7 + 12x^6 - 12x^5 + 12x^4 - 8x^3 + 5x^2 - 7x + 4.$$

Answer: $\boxed{-\frac{7}{16}}$ For convenience denote the polynomial by $P(x)$. Notice $4 + 8 = 7 + 5 = 12$ and that the consecutive terms $12x^6 - 12x^5 + 12x^4$ are the leading terms of $12\Phi_{14}(x)$, which is suggestive. Indeed, consider ω a primitive 14-th root of unity; since $\omega^7 = -1$, we have $4\omega^{10} = -4\omega^3$, $-7\omega^9 = 7\omega^2$, and so on, so that

$$P(\omega) = 12(\omega^6 - \omega^5 + \cdots + 1) = 12\Phi_{14}(\omega) = 0.$$

Dividing, we find

$$P(x) = \Phi_{14}(x)(4x^4 - 3x^3 - 2x^2 - 3x + 4).$$

This second polynomial is symmetric; since 0 is clearly not a root, we have

$$4x^4 - 3x^3 - 2x^2 - 3x + 4 = 0 \iff 4(x + \frac{1}{x})^2 - 3(x + \frac{1}{x}) - 10 = 0.$$

Setting $y = x + 1/x$ and solving the quadratic gives $y = 2$ and $y = -5/4$ as solutions; replacing y with $x + 1/x$ and solving the two resulting quadratics give the double root $x = 1$ and the roots $(-5 \pm i\sqrt{39})/8$ respectively. Together with the primitive fourteenth roots of unity, these are all the roots of our polynomial.

Explicitly, the roots are

$$e^{\pi i/7}, e^{3\pi i/7}, e^{5\pi i/7}, e^{9\pi i/7}, e^{11\pi i/7}, e^{13\pi i/7}, 1, (-5 \pm i\sqrt{39})/8.$$

The sum of squares of the roots of unity (including 1) is just 0 by symmetry (or a number of other methods). The sum of the squares of the final conjugate pair is $\frac{2(5^2-39)}{8^2} = -\frac{14}{32} = -\frac{7}{16}$.

31. [**20**] Define a *power cycle* to be a set S consisting of the nonnegative integer powers of an integer a, i.e. $S = \{1, a, a^2, \dots\}$ for some integer a. What is the minimum number of power cycles required such that given any odd integer n, there exists some integer k in one of the power cycles such that $n \equiv k \pmod{1024}$?

Answer: $\boxed{10}$ **Solution 1.** Partition the odd residues mod 1024 into 10 classes:

- Class 1: $1 \pmod 4$.
- Class n $(2 \le n \le 9)$: $2^n - 1 \pmod{2^{n+1}}$.
- Class 10: $-1 \pmod{1024}$.

Let S_a be the power cycle generated by a. If a is in class 1, all of S_a is in class 1. If a is in class n $(2 \le n \le 9)$, then S_a is in the union of class n and the residues $1 \pmod{2^{n+1}}$. If a is in class 10, then S_a is in the union of class n and the residues $1 \pmod{1024}$. Therefore, S_a cannot contain two of the following residues: $5, 2^2 - 1, 2^3 - 1, \dots 2^{10} - 1$, and that at least 10 cycles are needed.

Note that $5^{128} - 1 = (5-1)(5+1)(5^2+1)\cdots(5^{64}+1)$ has exactly 9 factors of 2 in its prime factorization, while $5^{256} - 1 = (5^{128} - 1)(5^{128} + 1)$ is divisible by 1024 so the order of 5 modulo 1024, the smallest positive power of 5 that is congruent to 1, is 256. Observe that among $5^0, 5^1, \dots 5^{255}$, the ratio between any two is a positive power of 5 smaller than 5^{256}, so the ratio is not congruent to 1 and any two terms are not congruent mod 1024. In addition, all terms are in class 1, and class 1 has 256 members, so S_5 contains members congruent to each element of class 1.

Similarly, let $2 \le n \le 9$. Then the order of a, where $a = 2^n - 1$, is 2^{10-n}. The 2^{9-n} terms $a^1, a^3, \dots a^{2^{10-n}-1}$ are pairwise not congruent and all in class n. Class n only has 2^{9-n} members, so S_a contains members congruent to each element of class n.

Finally, S_{-1} contains members congruent to the element of class 10.

The cycles S_5, S_{-1}, and 8 cycles S_a cover all the residues mod 1024, so the answer is 10.

Solution 2. Lemma. Given a positive integer $n \ge 3$, there exists an odd integer x such that the order of x modulo 2^n is 2^{n-2}.

Proof. We apply induction on n. The base cases of $n = 3, 4$ are clearly true with $x = 3$, so suppose we have the statement holds for $n-1$ and we wish to show it for n where $n \ge 5$. Suppose no such integer x exists, so we have $x^{2^{n-3}} \equiv 1 \pmod{2^n}$ for all odd x. But then remark that $(x^{2^{n-4}}-1)(x^{2^{n-4}}+1) \equiv 0 \pmod{2^n}$. As $n-4 \ge 1$ we have $x^{2^{n-4}}+1 \equiv 2 \pmod 4$ as all squares are $1 \pmod 4$, so it follows for the above relation to be true we require 2^{n-1} divides $x^{2^{n-4}}-1$ for all odd x. However, by taking x to have order 2^{n-3} modulo 2^{n-1} (which exists by the inductive hypothesis) we get a contradiction so we are done. $\square$

Now, let x have order 2^8 modulo 2^{10}. Remark that if for some integer k we had $x^k \equiv -1 \pmod{2^{10}}$, we would have $x^{2k} \equiv 1 \pmod{2^{10}}$ so $2^7 \mid k$. In that case k is even so as x is odd we have $x^k \equiv 1 \pmod 4$ but $-1 + 1024m$ is never $1 \pmod 4$ for any integer m so it follows -1 is not equal to x^k modulo 1024 for any integer k. Thus it follows that when we let S to be the set of powers of x, then no two elements in S and $-S$ are congruent modulo 1024. As the order of x is 256 and there are 512 possible odd remainders upon dividing by 1024, it immediately follows that every integer x is congruent modulo 1024 to $\pm x^k$ for some $1 \le k \le 256$ and some choice of sign.

Now, it is easy so see that $x, -x, -x^2, -x^4, \dots, -x^{2^7}, -x^{2^8}$ generating 10 power cycles works (x^a for any a is in the first, and then $-x^{2^k a}$ for odd a is in the power cycle generated by $-x^{2^k}$). To show that we cannot do any better suppose there exist 9 power cycles which include all odd integers modulo 1024. Then remark that $-x^{2^k}$ is congruent to a number in a power cycle modulo 1024 if and only if the power cycle is generated by an integer congruent to $x^{-2^k \cdot a} \pmod{1024}$ where a is an odd integer. It follows that 9 integers must be congruent to $-x^{a_1}, -x^{2a_2}, \dots, -x^{256a_9} \pmod{1024}$ for some odd integers $a_1, a_2, \dots, a_9$. But then 3 is clearly not in any of the power cycles, contradiction so it follows we must have at least 10 power cycles so we are done.

32. [**20**] A wealthy king has his blacksmith fashion him a large cup, whose inside is a cone of height 9 inches and base diameter 6 inches (that is, the opening at the top of the cup is 6 inches in diameter). At one of his many feasts, he orders the mug to be filled to the brim with cranberry juice.

For each positive integer n, the king stirs his drink vigorously and takes a sip such that the height of fluid left in his cup after the sip goes down by $\frac{1}{n^2}$ inches. Shortly afterwards, while the king is distracted, the court jester adds pure Soylent to the cup until it's once again full. The king takes sips precisely every minute, and his first sip is exactly one minute after the feast begins.

As time progresses, the amount of juice consumed by the king (in cubic inches) approaches a number r. Find r.

Answer: $\boxed{\frac{216\pi^3 - 2187\sqrt{3}}{8\pi^2}}$ First, we find the total amount of juice consumed. We can simply subtract the amount of juice remaining at infinity from the initial amount of juice in the cup, which of course is simply the volume of the cup; we'll denote this value by V.

Since volume in the cup varies as the cube of height, the amount of juice remaining in the cup after m minutes is

$$V \cdot \prod_{n=1}^{m} \left(\frac{9 - \frac{1}{n^2}}{9} \right)^3 = V \cdot \left(\prod_{n=1}^{m} \left(1 - \frac{1}{9n^2} \right) \right)^3.$$

We can now factor the term inside the product to find

$$V\left(\prod_{n=1}^{m}\frac{(3n+1)(3n-1)}{9n^2}\right)^3 = V\left(\frac{(3m+1)!}{3^{3m}(m!)^3}\right)^3.$$

If remains to evaluate the limit of this expression as m goes to infinity.

However, by Stirling's approximation, we have

$$\begin{aligned}\lim_{m\to\infty}\frac{(3m+1)!}{3^{3m}(m!)^3} &= \lim_{m\to\infty}\frac{(\frac{3n+1}{e})^{3n+1}\cdot\sqrt{2\pi(3n+1)}}{(\frac{3n}{e})^{3n}\sqrt{(2\pi n)^3}}\\ &= \lim_{m\to\infty}\frac{(3n+1)\sqrt{3}}{2\pi n e}\left(\frac{3n+1}{3n}\right)^{3n}\\ &= \frac{3\sqrt{3}}{2\pi}.\end{aligned}$$

Therefore the total amount of juice the king consumes is

$$V - V\left(\frac{3\sqrt{3}}{2\pi}\right)^3 = \left(\frac{3^2\cdot\pi\cdot 9}{3}\right)\left(\frac{8\pi^3-81\sqrt{3}}{8\pi^3}\right) = \frac{216\pi^3-2187\sqrt{3}}{8\pi^2}.$$

Remark. We present another way to calculate the limit at $m\to\infty$ of $f(m) = \frac{(3m+1)!}{3^{3m}(m!)^3}$. We have

$$f(m+1) = \frac{(3m+4)!}{3^{3m+3}(m+1)!^3} = f(m)\frac{(m+\frac{2}{3})(m+\frac{4}{3})}{(m+1)^2},$$

whence we can write

$$f(m) = \frac{c\Gamma(m+\frac{2}{3})\Gamma(m+\frac{4}{3})}{\Gamma(m+1)^2}$$

for some constant c. We can find c by equating the expressions at $m=0$; we have

$$1 = f(0) = \frac{c\Gamma(\frac{2}{3})\Gamma(\frac{4}{3})}{\Gamma(1)^2},$$

so that $c = \Gamma(1)^2/\Gamma(\frac{2}{3})\Gamma(\frac{4}{3})$.

Of course, $\Gamma(1) = 0! = 1$. We can evaluate the other product as follows:

$$\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right) = \frac{1}{3}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right) = \frac{1}{3}\cdot\frac{\pi}{\sin\pi/3} = \frac{2\pi}{3\sqrt{3}}.$$

Here the first step follows from $\Gamma(n+1) = n\Gamma(n)$, while the second follows from Euler's reflection formula. Thus $c = 3\sqrt{3}/2\pi$. We can now compute

$$\lim_{m\to\infty} f(m) = \lim_{m\to\infty}\frac{c\Gamma(m+\frac{2}{3})\Gamma(m+\frac{4}{3})}{\Gamma(m+1)^2} = \frac{3\sqrt{3}}{2\pi}\lim_{m\to\infty}\frac{\Gamma(m+\frac{2}{3})\Gamma(m+\frac{4}{3})}{\Gamma(m+1)^2}.$$

Since $\lim_{n\to\infty}\Gamma(n+\alpha)/[\Gamma(n)n^\alpha] = 1$, this final limit is 1 and $f(m)\to 3\sqrt{3}/2\pi$ as $m\to\infty$.

33. [**25**] Let N denote the sum of the decimal digits of $\binom{1000}{100}$. Estimate the value of N. If your answer is a positive integer A written fully in decimal notation (for example, 521495223), your score will be the greatest integer not exceeding $25\cdot(0.99)^{|A-N|}$. Otherwise, your score will be zero.

Answer: $\boxed{621}$ `http://www.wolframalpha.com/input/?i=sum+of+digits+of+nCr(1000,100)`

To see this, one can estimate there are about 150 digits, and we expect the digits to be roughly random, for $150\cdot 4.5\approx 675$, which is already very close to the actual answer. The actual number of digits is 140, and here $140\cdot 4.5 = 630$ is within 9 of the actual answer.

34. [**25**] For an integer n, let $f(n)$ denote the number of pairs (x, y) of integers such that $x^2+xy+y^2 = n$. Compute the sum

$$\sum_{n=1}^{10^6} nf(n).$$

Write your answer in the form $a \cdot 10^b$, where b is an integer and $1 \leq a < 10$ is a decimal number.

If your answer is written in this form, your score will be $\max\{0, 25 - \lfloor 100|\log_{10}(A/N)|\rfloor\}$, where $N = a \cdot 10^b$ is your answer to this problem and A is the actual answer. Otherwise, your score will be zero.

Answer: $\boxed{1.813759629294 \cdot 10^{12}}$ Rewrite the sum as

$$\sum_{x^2+xy+y^2 \leq 10^6} (x^2+xy+y^2),$$

where the sum is over all pairs (x, y) of integers with $x^2 + xy + y^2 \leq 10^6$. We can find a crude upper bound for this sum by noting that

$$x^2+xy+y^2 = \frac{3}{4}x^2 + \left(\frac{x}{2}+y\right)^2 \geq \frac{3}{4}x^2,$$

so each term of this sum has $|x| \leq \frac{2}{\sqrt{3}}10^3$. Similarly, $|y| \leq \frac{2}{\sqrt{3}}10^3$. Therefore, the number of terms in the sum is at most

$$\left(\frac{4}{\sqrt{3}}10^3+1\right)^2 \approx 10^6.$$

(We are throwing away "small" factors like $\frac{16}{3}$ in the approximation.) Furthermore, each term in the sum is at most 10^6, so the total sum is less than about 10^{12}. The answer $1 \cdot 10^{12}$ would unfortunately still get a score of 0.

For a better answer, we can approximate the sum by an integral:

$$\sum_{x^2+xy+y^2 \leq 10^6} (x^2+xy+y^2) \approx \iint_{x^2+xy+y^2 \leq 10^6} (x^2+xy+y^2)\, dy\, dx.$$

Performing the change of variables $(u, v) = \left(\frac{\sqrt{3}}{2}x, \frac{1}{2}x+y\right)$ and then switching to polar coordinates $(r, \theta) = (\sqrt{u^2+v^2}, \tan^{-1}(v/u))$ yields

$$\begin{aligned}\iint_{x^2+xy+y^2 \leq 10^6} (x^2+xy+y^2)\, dy\, dx &= \frac{2}{\sqrt{3}} \iint_{u^2+v^2 \leq 10^6} (u^2+v^2)\, dv\, du \\ &= \frac{2}{\sqrt{3}} \int_0^{2\pi} \int_0^{10^3} r^3\, dr\, d\theta \\ &= \frac{4\pi}{\sqrt{3}} \int_0^{10^3} r^3\, dr \\ &= \frac{\pi}{\sqrt{3}} \cdot 10^{12}.\end{aligned}$$

This is approximately $1.8138 \cdot 10^{12}$, which is much closer to the actual answer. (An answer of $1.8 \cdot 10^{12}$ is good enough for full credit.)

The answer can also be computed exactly by the Common Lisp code:

```
(defconstant +MAX+ 1e6)
(defvar +lower+ -2000)
(defvar +upper+ 2000)
(princ
```

```
(loop for x from +lower+ to +upper+ sum
      (loop for y from +lower+ to +upper+
            sum
            (let ((S (+ (* x x) (* x y) (* y y))))
              (if (and (<= S +MAX+) (> S 0)) S 0)))))
```

35. [**25**] Let P denote the set of all subsets of $\{1, \ldots, 23\}$. A subset $S \subseteq P$ is called *good* if whenever A, B are sets in S, the set $(A \setminus B) \cup (B \setminus A)$ is also in S. (Here, $A \setminus B$ denotes the set of all elements in A that are not in B, and $B \setminus A$ denotes the set of all elements in B that are not in A.) What **fraction** of the good subsets of P have between 2015 and 3015 elements, inclusive?

If your answer is a decimal number or a fraction (of the form m/n, where m and n are positive integers), then your score on this problem will be equal to $\max\{0, 25 - \lfloor 1000|A - N| \rfloor\}$, where N is your answer and A is the actual answer. Otherwise, your score will be zero.

Answer: $\boxed{\frac{18839183877670041942218307147122500601235}{47691684840486192422095701784512492731212} \approx 0.3950203047068107}$ Let $n = 23$, and $\ell = \lfloor n/2 \rfloor = 11$.

We use the well-known rephrasing of the symmetric difference $((A \setminus B) \cup (B \setminus A))$ in terms of addition modulo 2 of "indicators/characteristic vectors". So we simply want the number $\binom{n}{\ell}_2$ of dimension ℓ subspaces of the $F := \mathbb{F}_2$-vector space $V := \mathbb{F}_2^n$. Indeed, good subsets of 2^d elements simply correspond to dimension d subspaces (in particular, good subsets can only have sizes equal to powers of $|F| = 2$, and 2^ℓ is the only power between 2015 and 3015, inclusive).

To do this, it's easier to first count the number of (ordered) tuples of ℓ linearly independent elements of V, and divide (to get the subspace count) by the number of (ordered) tuples of ℓ linearly independent elements of any ℓ-dimensional subspace of V (a well-defined number independent of the choice of subspace).

In general, if we want to count tuples of m linearly independent elements in an n-dimensional space (with $n \geq m$), just note that we are building on top of (0) (the zero-dimensional subspace), and once we've chosen $r \leq m$ elements (with $0 \leq r \leq m-1$), there are $2^n - 2^r$ elements linearly independent to the previous r elements (which span a subspace of dimension r, hence of 2^r "bad" elements). Thus the number of m-dimensional subspaces of an n-dimensional space is

$$\binom{n}{m}_2 := \frac{(2^n - 2^0)(2^n - 2^1) \cdots (2^n - 2^{m-1})}{(2^m - 2^0)(2^m - 2^1) \cdots (2^m - 2^{m-1})},$$

a "Gaussian binomial coefficient."

We want to estimate

$$\frac{\binom{n}{\ell}_2}{\sum_{m=0}^{n} \binom{n}{m}_2} = \frac{1}{2} \frac{\binom{23}{11}_2}{\sum_{m=0}^{11} \binom{23}{m}_2}.$$

To do this, note that $\binom{n}{m}_2 = \binom{n}{n-m}_2$, so we may restrict our attention to the lower half. Intuitively, the Gaussian binomial coefficients should decay exponentially (or similarly quickly) away from the center; indeed, if $m \leq n/2$, then

$$\binom{n}{m-1}_2 \Big/ \binom{n}{m}_2 = \frac{(2^m - 2^0) \cdot 2^{m-1}}{(2^n - 2^{m-1})} \approx 2^{2m-1-n}.$$

So in fact, the decay is super-exponential, starting (for $n = 23$ odd and $m \leq \ell = 11$) at an $\approx \frac{1}{4}$ rate. So most of the terms (past the first 2 to 4, say) are negligible in our estimation. If we use the first two terms, we get an approximation of $\frac{1}{2} \cdot \frac{1}{1+\frac{1}{4}} = \frac{2}{5} = 0.4$, which is enough for 20 points. (Including the next term gives an approximation of $\frac{32}{81} \approx 0.3950617$, which is good enough to get full credit.)

To compute the exact answer, we used the following python3 code:

```
from functools import lru_cache
from fractions import Fraction

@lru_cache(maxsize=None)
def gauss_binom(n, k, e):
    if k < 0 or k > n:
        return 0
    if k == 0 or k == n:
        return 1
    return e ** k * gauss_binom(n - 1, k, e) + \
           gauss_binom(n - 1, k - 1, e)

N = 23
K = 11
good = gauss_binom(N, K, 2)
total = sum(gauss_binom(N, i, 2) for i in range(N + 1))
print(Fraction(good, total))
print(float(good/total))
```

36. [**25**] A prime number p is *twin* if at least one of $p+2$ or $p-2$ is prime and *sexy* if at least one of $p+6$ and $p-6$ is prime.

How many sexy twin primes (i.e. primes that are both twin and sexy) are there less than 10^9? Express your answer as a positive integer N in decimal notation; for example, 521495223. If your answer is in this form, your score for this problem will be $\max\left\{0, 25 - \lfloor \frac{1}{10000}|A-N| \rfloor\right\}$, where A is the actual answer to this problem. Otherwise, your score will be zero.

Answer: $\boxed{1462105}$ The Hardy-Littlewood conjecture states that given a set A of integers, the number of integers x such that $x+a$ is a prime for all $a \in A$ is

$$\frac{x}{(\ln x)^{|A|}} \prod_p \frac{1 - \frac{w(p;A)}{p}}{(1-\frac{1}{p})^k}(1+o(1))$$

where $w(p;A)$ is the number of distinct residues of A modulo p and the $o(1)$ term goes to 0 as x goes to infinity. Note that for the 4 tuples of the form $(0, \pm 2, \pm 6)$, $w(p;A) = 3$, and using the approximation $\frac{1-k/p}{(1-1/p)^k} \approx 1 - \binom{k}{2}/p^2 \approx (1 - \frac{1}{p^2})^{\binom{k}{2}}$, we have

$$\prod_{p>3} \frac{1-\frac{k}{p}}{(1-\frac{1}{p})^k} \approx \left(\frac{6}{\pi^2}\right)^{\binom{k}{2}} \cdot \left(\frac{4}{3}\right)^{\binom{k}{2}} \left(\frac{9}{8}\right)^{\binom{k}{2}} \approx \left(\frac{9}{10}\right)^{\binom{k}{2}}$$

Applying this for the four sets $A = (0, \pm 2, \pm 6)$, $x = 10^9$ (and approximating $\ln x = 20$ and just taking the $p = 2$ and $p = 3$ terms, we get the approximate answer

$$4 \cdot \frac{10^9}{20^3} \frac{1-\frac{1}{2}}{(\frac{1}{2})^3} \frac{1-\frac{2}{3}}{(\frac{1}{3})^3} \left(\frac{9}{10}\right)^3 = 1640250$$

One improvement we can make is to remove the double-counted tuples, in particular, integers x such that $x, x+6, x-6$, and one of $x \pm 2$ is prime. Again by the Hardy-Littlewood conjecture, the number of such x is approximately (using the same approximations)

$$2 \cdot \frac{10^9}{20^4} \frac{1-\frac{1}{2}}{(\frac{1}{2})^4} \frac{1-\frac{2}{3}}{(\frac{1}{3})^4} \left(\frac{9}{10}\right)^6 \approx 90000$$

Subtracting gives an estimate of about 1550000. Note that this is still an overestimate, as $\ln 10^9$ is actually about 20.7 and $\frac{1-k/p}{(1-1/p)^k} < (1-\frac{1}{p^2})^{\binom{k}{2}}$.

Here is the C++ code that we used to generate the answer:

```
#include<iostream>
#include<cstring> // memset
using namespace std;

const int MAXN = 1e9;
bool is_prime[MAXN + 6];

int main(){
    // Sieve of Eratosthenes
    memset(is_prime, true, sizeof(is_prime));
    is_prime[0] = is_prime[1] = false;
    for (int i=2; i<MAXN + 6; i++){
        if (is_prime[i]){
            for (int j=2 * i; j < MAXN + 6; j += i){
                is_prime[j] = false;
            }
        }
    }

    // Count twin sexy primes.
    int ans = 1; // 5 is the only twin sexy prime < 6.
    for (int i=6; i<MAXN; i++){
        if (is_prime[i]
            && (is_prime[i-6] || is_prime[i+6])
            && (is_prime[i-2] || is_prime[i+2])) {
             ans++;
        }
    }

    cout << ans << endl;

    return 0;
}
```