HMMT November 2015

November 14, 2015

Individual

1. Find the number of triples (a, b, c) of positive integers such that $a + ab + abc = 11$.

2. Let a and b be real numbers randomly (and independently) chosen from the range $[0, 1]$. Find the probability that a, b and 1 form the side lengths of an obtuse triangle.

3. Neo has an infinite supply of red pills and blue pills. When he takes a red pill, his weight will double, and when he takes a blue pill, he will lose one pound. If Neo originally weighs one pound, what is the minimum number of pills he must take to make his weight 2015 pounds?

4. Chords AB and CD of a circle are perpendicular and intersect at a point P. If $AP = 6, BP = 12$, and $CD = 22$, find the area of the circle.

5. Let S be a subset of the set $\{1, 2, 3, \ldots, 2015\}$ such that for any two elements $a, b \in S$, the difference $a - b$ does not divide the sum $a + b$. Find the maximum possible size of S.

6. Consider all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f(f(x) + 2x + 20) = 15.$$

   Call an integer n *good* if $f(n)$ can take any integer value. In other words, if we fix n, for any integer m, there exists a function f such that $f(n) = m$. Find the sum of all good integers x.

7. Let $\triangle ABC$ be a right triangle with right angle C. Let I be the incenter of ABC, and let M lie on AC and N on BC, respectively, such that M, I, N are collinear and $\overline{MN}$ is parallel to AB. If $AB = 36$ and the perimeter of CMN is 48, find the area of ABC.

8. Let $ABCD$ be a quadrilateral with an inscribed circle ω that has center I. If $IA = 5, IB = 7, IC = 4, ID = 9$, find the value of $\frac{AB}{CD}$.

9. Rosencrantz plays $n \leq 2015$ games of question, and ends up with a win rate $\left(\text{i.e. } \frac{\text{\# of games won}}{\text{\# of games played}}\right)$ of k. Guildenstern has also played several games, and has a win rate less than k. He realizes that if, after playing some more games, his win rate becomes higher than k, then there must have been some point in time when Rosencrantz and Guildenstern had the exact same win-rate. Find the product of all possible values of k.

10. Let N be the number of functions f from $\{1, 2, \ldots, 101\} \to \{1, 2, \ldots, 101\}$ such that $f^{101}(1) = 2$. Find the remainder when N is divided by 103.