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1. [**5**] Farmer Yang has a 2015×2015 square grid of corn plants. One day, the plant in the very center of the grid becomes diseased. Every day, every plant adjacent to a diseased plant becomes diseased. After how many days will all of Yangs corn plants be diseased?

2. [**5**] The three sides of a right triangle form a geometric sequence. Determine the ratio of the length of the hypotenuse to the length of the shorter leg.

3. [**5**] A parallelogram has 2 sides of length 20 and 15. Given that its area is a positive integer, find the minimum possible area of the parallelogram.

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4. [**6**] Eric is taking a biology class. His problem sets are worth 100 points in total, his three midterms are worth 100 points each, and his final is worth 300 points. If he gets a perfect score on his problem sets and scores $60\%, 70\%$, and 80% on his midterms respectively, what is the minimum possible percentage he can get on his final to ensure a passing grade? (Eric passes if and only if his overall percentage is at least 70%).

5. [**6**] James writes down three integers. Alex picks some two of those integers, takes the average of them, and adds the result to the third integer. If the possible final results Alex could get are 42, 13, and 37, what are the three integers James originally chose?

6. [**6**] Let AB be a segment of length 2 with midpoint M. Consider the circle with center O and radius r that is externally tangent to the circles with diameters AM and BM and internally tangent to the circle with diameter AB. Determine the value of r.

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7. [**7**] Let n be the smallest positive integer with exactly 2015 positive factors. What is the sum of the (not necessarily distinct) prime factors of n? For example, the sum of the prime factors of 72 is $2 + 2 + 2 + 3 + 3 = 14$.

8. [**7**] For how many pairs of nonzero integers (c, d) with $-2015 \leq c, d \leq 2015$ do the equations $cx = d$ and $dx = c$ both have an integer solution?

9. [**7**] Find the smallest positive integer n such that there exists a complex number z, with positive real and imaginary part, satisfying $z^n = (\overline{z})^n$.

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10. [**8**] Call a string of letters S an *almost palindrome* if S and the reverse of S differ in exactly two places. Find the number of ways to order the letters in $HMMTTHEMETEAM$ to get an almost palindrome.

11. [**8**] Find all integers n, not necessarily positive, for which there exist positive integers a, b, c satisfying $a^n + b^n = c^n$.

12. [**8**] Let a and b be positive real numbers. Determine the minimum possible value of

$$\sqrt{a^2+b^2}+\sqrt{(a-1)^2+b^2}+\sqrt{a^2+(b-1)^2}+\sqrt{(a-1)^2+(b-1)^2}$$

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13. [**9**] Consider a 4×4 grid of squares, each of which are originally colored red. Every minute, Piet can jump on one of the squares, changing the color of it and any adjacent squares (two squares are adjacent if they share a side) to blue. What is the minimum number of minutes it will take Piet to change the entire grid to blue?

14. [**9**] Let ABC be an acute triangle with orthocenter H. Let D, E be the feet of the A, B-altitudes respectively. Given that $AH = 20$ and $HD = 15$ and $BE = 56$, find the length of BH.

15. [**9**] Find the smallest positive integer b such that 1111_b (1111 in base b) is a perfect square. If no such b exists, write "No solution".

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16. [**10**] For how many triples (x, y, z) of integers between -10 and 10 inclusive do there exist reals a, b, c that satisfy

$$ab = x$$
$$ac = y$$
$$bc = z?$$

17. [**10**] Unit squares $ABCD$ and $EFGH$ have centers O_1 and O_2 respectively, and are originally situated such that B and E are at the same position and C and H are at the same position. The squares then rotate clockwise about their centers at the rate of one revolution per hour. After 5 minutes, what is the area of the intersection of the two squares?

18. [**10**] A function f satisfies, for all nonnegative integers x and y:

- $f(0, x) = f(x, 0) = x$
- If $x \geq y \geq 0$, $f(x, y) = f(x - y, y) + 1$
- If $y \geq x \geq 0$, $f(x, y) = f(x, y - x) + 1$

Find the maximum value of f over $0 \leq x, y \leq 100$.

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19. [**11**] Each cell of a 2×5 grid of unit squares is to be colored white or black. Compute the number of such colorings for which no 2×2 square is a single color.

20. [**11**] Let n be a three-digit integer with nonzero digits, not all of which are the same. Define $f(n)$ to be the greatest common divisor of the six integers formed by any permutation of ns digits. For example, $f(123) = 3$, because $\gcd(123, 132, 213, 231, 312, 321) = 3$. Let the maximum possible value of $f(n)$ be k. Find the sum of all n for which $f(n) = k$.

21. [**11**] Consider a 2×2 grid of squares. Each of the squares will be colored with one of 10 colors, and two colorings are considered equivalent if one can be rotated to form the other. How many distinct colorings are there?

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22. [**12**] Find all the roots of the polynomial $x^5 - 5x^4 + 11x^3 - 13x^2 + 9x - 3$.

23. [**12**] Compute the smallest positive integer n for which

$$0 < \sqrt[4]{n} - \lfloor \sqrt[4]{n} \rfloor < \frac{1}{2015}.$$

24. [**12**] Three ants begin on three different vertices of a tetrahedron. Every second, they choose one of the three edges connecting to the vertex they are on with equal probability and travel to the other vertex on that edge. They all stop when any two ants reach the same vertex at the same time. What is the probability that all three ants are at the same vertex when they stop?

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25. [**13**] Let ABC be a triangle that satisfies $AB = 13, BC = 14, AC = 15$. Given a point P in the plane, let P_A, P_B, P_C be the reflections of A, B, C across P. Call P *good* if the circumcircle of $P_A P_B P_C$ intersects the circumcircle of ABC at exactly 1 point. The locus of good points P encloses a region $\mathcal{S}$. Find the area of $\mathcal{S}$.

26. [**13**] Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a *continuous* function satisfying $f(xy) = f(x) + f(y) + 1$ for all positive reals x, y. If $f(2) = 0$, compute $f(2015)$.

27. [**13**] Let $ABCD$ be a quadrilateral with $A = (3, 4), B = (9, -40), C = (-5, -12), D = (-7, 24)$. Let P be a point in the plane (not necessarily inside the quadrilateral). Find the minimum possible value of $AP + BP + CP + DP$.

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28. [**15**] Find the shortest distance between the lines $\frac{x+2}{2} = \frac{y-1}{3} = \frac{z}{1}$ and $\frac{x-3}{-1} = \frac{y}{1} = \frac{z+1}{2}$

29. [**15**] Find the largest real number k such that there exists a sequence of positive reals $\{a_i\}$ for which $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n^k}$ does not.

30. [**15**] Find the largest integer n such that the following holds: there exists a set of n points in the plane such that, for any choice of three of them, some two are unit distance apart.

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31. [**17**] Two random points are chosen on a segment and the segment is divided at each of these two points. Of the three segments obtained, find the probability that the largest segment is more than three times longer than the smallest segment.

32. [**17**] Find the sum of all positive integers $n \leq 2015$ that can be expressed in the form $\lceil \frac{x}{2} \rceil + y + xy$, where x and y are positive integers.

33. [**17**] How many ways are there to place four points in the plane such that the set of pairwise distances between the points consists of exactly 2 elements? (Two configurations are the same if one can be obtained from the other via rotation and scaling.)

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34. [**20**] Let n be the **second** smallest integer that can be written as the sum of two positive cubes in two different ways. Compute n. If your guess is a, you will receive $\max\left(25 - 5 \cdot \max\left(\frac{a}{n}, \frac{n}{a}\right), 0\right)$ points, rounded up.

35. [**20**] Let n be the smallest positive integer such that any positive integer can be expressed as the sum of n integer 2015th powers. Find n. If your answer is a, your score will be $\max(20 - \frac{1}{5}|\log_{10} \frac{a}{n}|, 0)$, rounded up.

36. [**20**] Consider the following seven false conjectures with absurdly high counterexamples. Pick any subset of them, and list their labels in order of their smallest counterexample (the smallest n for which the conjecture is false) from smallest to largest. For example, if you believe that the below list is already ordered by counterexample size, you should write "PECRSGA".

- P. (**Polya's conjecture**) For any integer n, at least half of the natural numbers below n have an odd number of prime factors.
- E. (**Euler's conjecture**) There is no perfect cube n that can be written as the sum of three positive cubes.
- C. (**Cyclotomic**) The polynomial with minimal degree whose roots are the primitive nth roots of unity has all coefficients equal to -1, 0, or 1.

- R. (**Prime race**) For any integer n, there are more primes below n equal to $2 \pmod 3$ than there are equal to $1 \pmod 3$.
- S. (**Seventeen conjecture**) For any integer n, $n^{17}+9$ and $(n+1)^{17}+9$ are relatively prime.
- G. (**Goldbach's (other) conjecture**) Any odd composite integer n can be written as the sum of a prime and twice a square.
- A. (**Average square**) Let $a_1 = 1$ and $a_{k+1} = \frac{1+a_1^2+a_2^2+\ldots+a_k^2}{k}$. Then a_n is an integer for any n.

If your answer is a list of $4 \le n \le 7$ labels in the correct order, your score will be $(n-2)(n-3)$. Otherwise, it will be 0.