HMMT November 2015

November 14, 2015

Guts round

1. [**5**] Farmer Yang has a 2015×2015 square grid of corn plants. One day, the plant in the very center of the grid becomes diseased. Every day, every plant adjacent to a diseased plant becomes diseased. After how many days will all of Yangs corn plants be diseased?

   Proposed by: Alexander Katz

   Answer: $\boxed{2014}$

   After k minutes, the diseased plants are the ones with taxicab distance at most k from the center. The plants on the corner are the farthest from the center and have taxicab distance 2014 from the center, so all the plants will be diseased after 2014 minutes.

2. [**5**] The three sides of a right triangle form a geometric sequence. Determine the ratio of the length of the hypotenuse to the length of the shorter leg.

   Proposed by: Alexander Katz

   Answer: $\boxed{\frac{1+\sqrt{5}}{2}}$

   Let the shorter leg have length ℓ, and the common ratio of the geometric sequence be $r > 1$. Then the length of the other leg is ℓr, and the length of the hypotenuse is ℓr^2. Hence,

$$\ell^2 + (\ell r)^2 = (\ell r^2)^2$$
$$\implies \ell^2(r^2 + 1) = \ell^2 r^4$$
$$\implies r^2 + 1 = r^4$$

   Hence, $r^4 - r^2 - 1 = 0$, and therefore $r^2 = \frac{1\pm\sqrt{5}}{2}$. As $r > 1$, we have $r^2 = \boxed{\dfrac{1+\sqrt{5}}{2}}$, completing the problem as the ratio of the hypotenuse to the shorter side is $\frac{\ell r^2}{\ell} = r^2$.

3. [**5**] A parallelogram has 2 sides of length 20 and 15. Given that its area is a positive integer, find the minimum possible area of the parallelogram.

   Proposed by: Yang Liu

   Answer: $\boxed{1}$

   The area of the parallelogram can be made arbitrarily small, so the smallest positive integer area is 1.

4. [**6**] Eric is taking a biology class. His problem sets are worth 100 points in total, his three midterms are worth 100 points each, and his final is worth 300 points. If he gets a perfect score on his problem sets and scores $60\%, 70\%$, and 80% on his midterms respectively, what is the minimum possible percentage he can get on his final to ensure a passing grade? (Eric passes if and only if his overall percentage is at least 70%).

   Proposed by: Alexander Katz

   Answer: $\boxed{60\%}$

   We see there are a total of $100 + 3 \times 100 + 300 = 700$ points, and he needs $70\% \times 700 = 490$ of them. He has $100 + 60 + 70 + 80 = 310$ points before the final, so he needs 180 points out of 300 on the final, which is 60%.

5. [**6**] James writes down three integers. Alex picks some two of those integers, takes the average of them, and adds the result to the third integer. If the possible final results Alex could get are 42, 13, and 37, what are the three integers James originally chose?

   Proposed by: Alexander Katz

Answer: $\boxed{-20, 28, 38}$

Let x, y, z be the integers. We have

$$\frac{x+y}{2} + z = 42$$
$$\frac{y+z}{2} + x = 13$$
$$\frac{x+z}{2} + y = 37$$

Adding these three equations yields $2(x+y+z) = 92$, so $\frac{x+y}{2} + z = 23 + \frac{z}{2} = 42$ so $z = 38$. Similarly, $x = -20$ and $y = 28$.

6. [**6**] Let AB be a segment of length 2 with midpoint M. Consider the circle with center O and radius r that is externally tangent to the circles with diameters AM and BM and internally tangent to the circle with diameter AB. Determine the value of r.

Proposed by: Sam Korsky

Answer: $\boxed{\frac{1}{3}}$

Let X be the midpoint of segment AM. Note that $OM \perp MX$ and that $MX = \frac{1}{2}$ and $OX = \frac{1}{2} + r$ and $OM = 1 - r$. Therefore by the Pythagorean theorem, we have

$$OM^2 + MX^2 = OX^2 \implies (1-r)^2 + \frac{1}{2^2} = \left(\frac{1}{2} + r\right)^2$$

which we can easily solve to find that $r = \boxed{\frac{1}{3}}$.

7. [**7**] Let n be the smallest positive integer with exactly 2015 positive factors. What is the sum of the (not necessarily distinct) prime factors of n? For example, the sum of the prime factors of 72 is $2 + 2 + 2 + 3 + 3 = 14$.

Proposed by: Alexander Katz

Answer: $\boxed{116}$

Note that $2015 = 5 \times 13 \times 31$ and that $N = 2^{30} \cdot 3^{12} \cdot 5^4$ has exactly 2015 positive factors. We claim this is the smallest such integer. Note that $N < 2^{66}$.

If n has 3 distinct prime factors, it must be of the form $p^{30}q^{12}r^4$ for some primes p, q, r, so $n \geq 2^{30} \cdot 3^{12} \cdot 5^4$.

If n has 2 distinct prime factors, it must be of the form $p^e q^f > 2^{e+f}$ where $(e+1)(f+1) = 2015$. It is easy to see that this means $e + f > 66$ so $n > 2^{66} > N$.

If n has only 1 prime factor, we have $n \geq 2^{2014} > N$.

So N is the smallest such integer, and the sum of its prime factors is $2 \cdot 30 + 3 \cdot 12 + 5 \cdot 4 = 116$.

8. [**7**] For how many pairs of nonzero integers (c, d) with $-2015 \leq c, d \leq 2015$ do the equations $cx = d$ and $dx = c$ both have an integer solution?

Proposed by: Yang Liu

Answer: $\boxed{8060}$

We need both c/d and d/c to be integers, which is equivalent to $|c| = |d|$, or $d = \pm c$. So there are 4030 ways to pick c and 2 ways to pick d, for a total of 8060 pairs.

9. [**7**] Find the smallest positive integer n such that there exists a complex number z, with positive real and imaginary part, satisfying $z^n = (\overline{z})^n$.

Proposed by: Alexander Katz

Since $|z| = |\overline{z}|$ we may divide by $|z|$ and assume that $|z| = 1$. Then $\overline{z} = \frac{1}{z}$, so we are looking for the smallest positive integer n such that there is a $2n^{\text{th}}$ root of unity in the first quadrant. Clearly there is a sixth root of unity in the first quadrant but no fourth or second roots of unity, so $n = \boxed{3}$ is the smallest.

10. [**8**] Call a string of letters S an *almost palindrome* if S and the reverse of S differ in exactly two places. Find the number of ways to order the letters in $HMMTTHEMETEAM$ to get an almost palindrome.

Proposed by: Yang Liu

Answer: $\boxed{2160}$

Note that T, E, A are used an odd number of times. Therefore, one must go in the middle spot and the other pair must match up. There are are $3 \cdot 2(\frac{6!}{2!}) = 2160$ ways to fill in the first six spots with the letters T, H, E, M, M and a pair of different letters. The factor of 3 accounts for which letter goes in the middle.

11. [**8**] Find all integers n, not necessarily positive, for which there exist positive integers a, b, c satisfying $a^n + b^n = c^n$.

Proposed by: Rikhav Shah

Answer: $\boxed{\pm 1, \pm 2}$

By Fermat's Last Theorem, we know $n < 3$. Suppose $n \leq -3$. Then $a^n + b^n = c^n \implies (bc)^{-n} + (ac)^{-n} = (ab)^{-n}$, but since $-n \geq 3$, this is also impossible by Fermat's Last Theorem. As a result, $|n| < 3$.

Furthermore, $n \neq 0$, as $a^0 + b^0 = c^0 \implies 1 + 1 = 1$, which is false. We now just need to find constructions for $n = -2, -1, 1, 2$. When $n = 1$, $(a, b, c) = (1, 2, 3)$ suffices, and when $n = 2$, $(a, b, c) = (3, 4, 5)$ works nicely. When $n = -1$, $(a, b, c) = (6, 3, 2)$ works, and when $n = -2$, $(a, b, c) = (20, 15, 12)$ is one example. Therefore, the working values are $n = \boxed{\pm 1, \pm 2}$.

12. [**8**] Let a and b be positive real numbers. Determine the minimum possible value of

$$\sqrt{a^2 + b^2} + \sqrt{(a-1)^2 + b^2} + \sqrt{a^2 + (b-1)^2} + \sqrt{(a-1)^2 + (b-1)^2}$$

Proposed by: Alexander Katz

Answer: $\boxed{2\sqrt{2}}$

Let $ABCD$ be a square with $A = (0, 0), B = (1, 0), C = (1, 1), D = (0, 1)$, and P be a point in the same plane as $ABCD$. Then the desired expression is equivalent to $AP + BP + CP + DP$. By the triangle inequality, $AP + CP \geq AC$ and $BP + DP \geq BD$, so the minimum possible value is $AC + BD = 2\sqrt{2}$. This is achievable when $a = b = \frac{1}{2}$, so we are done.

13. [**9**] Consider a 4×4 grid of squares, each of which are originally colored red. Every minute, Piet can jump on one of the squares, changing the color of it and any adjacent squares (two squares are adjacent if they share a side) to blue. What is the minimum number of minutes it will take Piet to change the entire grid to blue?

Proposed by: Alexander Katz

Answer: $\boxed{4}$

Piet can change the colors of at most 5 squares per minute, so as there are 16 squares, it will take him at least four minutes to change the colors of every square. Some experimentation yields that it is indeed possible to make the entire grid blue after 4 minutes; one example is shown below:

	X		
			X
X			
		X	

Here, jumping on the squares marked with an X provides the desired all-blue grid.

14. [**9**] Let ABC be an acute triangle with orthocenter H. Let D, E be the feet of the A, B-altitudes respectively. Given that $AH = 20$ and $HD = 15$ and $BE = 56$, find the length of BH.

Proposed by: Sam Korsky

Answer: $\boxed{50}$

Let x be the length of BH. Note that quadrilateral $ABDE$ is cyclic, so by Power of a Point, $x(56-x) = 20 \cdot 15 = 300$. Solving for x, we get $x = 50$ or 6. We must have $BH > HD$ so $x = 50$ is the correct length.

15. [**9**] Find the smallest positive integer b such that 1111_b (1111 in base b) is a perfect square. If no such b exists, write "No solution".

Proposed by: Alexander Katz

Answer: $\boxed{7}$

We have $1111_b = b^3 + b^2 + b + 1 = (b^2+1)(b+1)$. Note that $\gcd(b^2+1, b+1) = \gcd(b^2+1-(b+1)(b-1), b+1) = \gcd(2, b+1)$, which is either 1 or 2. If the gcd is 1, then there is no solution as this implies b^2+1 is a perfect square, which is impossible for positive b. Hence the gcd is 2, and $b^2+1, b+1$ are both twice perfect squares.

Let $b+1 = 2a^2$. Then $b^2+1 = (2a^2-1)^2+1 = 4a^4-4a^2+2 = 2(2a^4-2a^2+1)$, so $2a^4-2a^2+1 = (a^2-1)^2+(a^2)^2$ must be a perfect square. This first occurs when $a^2-1 = 3, a^2 = 4 \implies a = 2$, and thus $b = \boxed{7}$. Indeed, $1111_7 = 20^2$.

16. [**10**] For how many triples (x, y, z) of integers between -10 and 10 inclusive do there exist reals a, b, c that satisfy

$$ab = x$$
$$ac = y$$
$$bc = z?$$

Proposed by: Yang Liu

Answer: $\boxed{4061}$

If none are of x, y, z are zero, then there are $4 \cdot 10^3 = 4000$ ways, since xyz must be positive. Indeed, $(abc)^2 = xyz$. So an even number of them are negative, and the ways to choose an even number of 3 variables to be negative is 4 ways. If one of x, y, z is 0, then one of a, b, c is zero at least. So at least two of x, y, z must be 0. If all 3 are zero, this gives 1 more solution. If exactly 2 are negative, then this gives $3 \cdot 20$ more solutions. This comes from choosing one of x, y, z to be nonzero, and choosing its value in 20 ways.

Our final answer is $4000 + 60 + 1 = 4061$.

17. [**10**] Unit squares $ABCD$ and $EFGH$ have centers O_1 and O_2 respectively, and are originally situated such that B and E are at the same position and C and H are at the same position. The squares then rotate clockwise about their centers at the rate of one revolution per hour. After 5 minutes, what is the area of the intersection of the two squares?

Proposed by: Alexander Katz

Answer: $\boxed{\frac{2-\sqrt{3}}{4}}$

Note that $AE = BF = CG = DH = 1$ at all times. Suppose that the squares have rotated θ radians. Then $\angle O_1O_2H = \frac{\pi}{4} - \theta = \angle O_1DH$, so $\angle HDC = \frac{\pi}{4} - \angle O_1DH = \theta$. Let P be the intersection of AB and EH and Q be the intersection of BC and GH. Then $PH \parallel BQ$ and $HQ \parallel PB$, and $\angle PHG = \frac{\pi}{2}$, so $PBQH$ - our desired intersection - is a rectangle. We have $BQ = 1 - QC = 1 - \sin\theta$ and $HQ = 1 - \cos\theta$, so our desired area is $(1-\cos\theta)(1-\sin\theta)$. After 5 minutes, we have $\theta = \frac{2\pi}{12} = \frac{\pi}{6}$, so our answer is $\boxed{\dfrac{2-\sqrt{3}}{4}}$.

18. [**10**] A function f satisfies, for all nonnegative integers x and y:

- $f(0, x) = f(x, 0) = x$
- If $x \geq y \geq 0$, $f(x, y) = f(x - y, y) + 1$
- If $y \geq x \geq 0$, $f(x, y) = f(x, y - x) + 1$

Find the maximum value of f over $0 \leq x, y \leq 100$.

Proposed by: Alexander Katz

Answer: $\boxed{101}$

Firstly, $f(100, 100) = 101$.

To see this is maximal, note that $f(x, y) \leq \max\{x, y\} + 1$, say by induction on $x + y$.

19. [**11**] Each cell of a 2×5 grid of unit squares is to be colored white or black. Compute the number of such colorings for which no 2×2 square is a single color.

Proposed by: Alexander Katz

Answer: $\boxed{634}$

Let a_n denote the number of ways to color a $2 \times n$ grid subject only to the given constraint, and b_n denote the number of ways to color a $2 \times n$ grid subject to the given constraint, but with the added restriction that the first column cannot be colored black-black.

Consider the first column of a $2 \times n$ grid that is not subject to the additional constraint. It can be colored black-white or white-black, in which case the leftmost 2x2 square is guaranteed not to be monochromatic, and so the remaining $2 \times (n-1)$ subgrid can be colored in a_{n-1} ways. Otherwise, it is colored white-white or black-black; WLOG, assume that it's colored black-black. Then the remaining $2 \times (n-1)$ subgrid is subject to both constraints, so there are b_{n-1} ways to color the remaining subgrid. Hence $a_n = 2a_{n-1} + 2b_{n-1}$.

Now consider the first column of a $2 \times n$ grid that is subject to the additional constraint. The first column cannot be colored black-black, and if it is colored white-black or black-white, there are a_{n-1} ways to color the remaining subgrid by similar logic to the previous case. If it is colored white-white, then there are b_{n-1} ways to color the remaining subgrid, again by similar logic to the previous case. Hence $b_n = 2a_{n-1} + b_{n-1}$.

Therefore, we have $b_n = 2a_{n-1} + \frac{1}{2}(a_n - 2a_{n-1})$, and so $a_n = 2a_{n-1} + 2b_{n-1} = 2a_{n-1} + 2(2a_{n-2} + \frac{1}{2}(a_{n-1} - 2a_{n-2})) = 3a_{n-1} + 2a_{n-2}$. Finally, we have $a_0 = 1$ (as the only possibility is to, well, do nothing) and $a_1 = 4$ (as any 2×1 coloring is admissible), so $a_2 = 14, a_3 = 50, a_4 = 178, a_5 = \boxed{634}$.

20. [**11**] Let n be a three-digit integer with nonzero digits, not all of which are the same. Define $f(n)$ to be the greatest common divisor of the six integers formed by any permutation of ns digits. For example, $f(123) = 3$, because $\gcd(123, 132, 213, 231, 312, 321) = 3$. Let the maximum possible value of $f(n)$ be k. Find the sum of all n for which $f(n) = k$.

Proposed by: Alexander Katz

Answer: $\boxed{5994}$

Let $n = \overline{abc}$, and assume without loss of generality that $a \geq b \geq c$. We have $k \mid 100a + 10b + c$ and $k \mid 100a + 10c + b$, so $k \mid 9(b - c)$. Analogously, $k \mid 9(a - c)$ and $k \mid 9(a - b)$. Note that if $9 \mid n$, then 9 also divides any permutation of ns digits, so $9 \mid f(n)$ as well; ergo, $f(n) \geq 9$, implying that $k \geq 9$. If k is not a multiple of 3, then we have $k \mid c - a \implies k \leq c - a < 9$, contradiction, so $3 \mid k$.

Let $x = \min(a - b, b - c, a - c)$. If $x = 1$, then we have $k \mid 9$, implying $k = 9$ - irrelevant to our investigation. So we can assume $x \geq 2$. Note also that $x \leq 4$, as $2x \leq (a - b) + (b - c) = a - c \leq 9 - 1$, and if $x = 4$ we have $n = 951 \implies f(n) = 3$. If $x = 3$, then since $3 \mid k \mid 100a + 10b + c \implies 3 \mid a + b + c$, we have $a \equiv b \equiv c \pmod 3$ (e.g. if $b - c = 3$, then $b \equiv c \pmod 3$, so $a \equiv b \equiv c \pmod 3$) - the other cases are analogous). This gives us the possibilites $n = 147, 258, 369$, which give $f(n) = 3, 3, 9$ respectively.

Hence we can conclude that $x = 2$; therefore $k \mid 18$. We know also that $k \geq 9$, so either $k = 9$ or $k = 18$. If $k = 18$, then all the digits of n must be even, and n must be a multiple of 9; it is clear that these are sufficient criteria. As n's digits are all even, the sum of them is also even, and hence their sum is 18. Since $a \geq b \geq c$, we have $a + b + c = 18 \leq 3a \implies a \geq 6$, but if $a = 6$ then $a = b = c = 6$, contradicting the problem statement. Thus $a = 8$, and this gives us the solutions $n = 882, 864$ along with their permutations.

It remains to calculate the sum of the permutations of these solutions. In the $n = 882$ case, each digit is either 8, 8, or 2 (one time each), and in the $n = 864$ case, each digit is either 8, 6, or 4 (twice each). Hence the desired sum is $111(8 + 8 + 2) + 111(8 \cdot 2 + 6 \cdot 2 + 4 \cdot 2) = 111(54) = \boxed{5994}$.

21. [**11**] Consider a 2×2 grid of squares. Each of the squares will be colored with one of 10 colors, and two colorings are considered equivalent if one can be rotated to form the other. How many distinct colorings are there?

Proposed by: Sam Korsky

Answer: $\boxed{2530}$

This solution will be presented in the general case with n colors. Our problem asks for $n = 10$.

We isolate three cases:

Case 1: Every unit square has the same color

In this case there are clearly n ways to color the square.

Case 2: Two non-adjacent squares are the same color, and the other two squares are also the same color (but not all four squares are the same color).

In this case there are clearly $\binom{n}{2} = \frac{n(n-1)}{2}$ ways to color the square.

Case 3: Every other case

Since without the "rotation" condition there would be n^4 colorings, we have that in this case by complementary counting there are $\frac{n^4 - n(n-1) - n}{4}$ ways to color the square.

Therefore the answer is

$$n + \frac{n^2 - n}{2} + \frac{n^4 - n^2}{4} = \frac{n^4 + n^2 + 2n}{4} = \boxed{2530}$$

22. [**12**] Find all the roots of the polynomial $x^5 - 5x^4 + 11x^3 - 13x^2 + 9x - 3$.

Proposed by: Sam Korsky

Answer: $\boxed{1, \frac{3+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}, \frac{3-\sqrt{3}i}{2}, \frac{1+\sqrt{3}i}{2}}$

The $x^5 - 5x^4$ at the beginning of the polynomial motivates us to write it as $(x-1)^5 + x^3 - 3x^2 + 4x - 2$ and again the presence of the $x^3 - 3x^2$ motivates writing the polynomial in the form $(x-1)^5 + (x-1)^3 + (x-1)$. Let a and b be the roots of the polynomial $x^2 + x + 1$. It's clear that the roots of our polynomial are given by 1 and the roots of the polynomials $(x-1)^2 = a$ and $(x-1)^2 = b$. The quadratic formula shows that WLOG $a = \frac{-1+\sqrt{3}i}{2}$ and $b = \frac{-1-\sqrt{3}i}{2}$ so we find that either $x - 1 = \pm\frac{1+\sqrt{3}i}{2}$ or $x - 1 = \pm\frac{1-\sqrt{3}i}{2}$. Hence our roots are $\boxed{1, \frac{3+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}, \frac{3-\sqrt{3}i}{2}, \frac{1+\sqrt{3}i}{2}}$.

23. [**12**] Compute the smallest positive integer n for which

$$0 < \sqrt[4]{n} - \lfloor \sqrt[4]{n} \rfloor < \frac{1}{2015}.$$

Proposed by: Alexander Katz

Answer: $\boxed{4097}$

Let $n = a^4 + b$ where a, b are integers and $0 < b < 4a^3 + 6a^2 + 4a + 1$. Then

$$\begin{aligned}
\sqrt[4]{n} - \lfloor \sqrt[4]{n} \rfloor &< \frac{1}{2015} \\
\sqrt[4]{a^4 + b} - a &< \frac{1}{2015} \\
\sqrt[4]{a^4 + b} &< a + \frac{1}{2015} \\
a^4 + b &< \left(a + \frac{1}{2015}\right)^4 \\
a^4 + b &< a^4 + \frac{4a^3}{2015} + \frac{6a^2}{2015^2} + \frac{4a}{2015^3} + \frac{1}{2015^4}
\end{aligned}$$

To minimize $n = a^4 + b$, we clearly should minimize b, which occurs at $b = 1$. Then

$$1 < \frac{4a^3}{2015} + \frac{6a^2}{2015^2} + \frac{4a}{2015^3} + \frac{1}{2015^4}.$$

If $a = 7$, then $\frac{6a^2}{2015^2}, \frac{4a}{2015^3}, \frac{1}{2015^4} < \frac{1}{2015}$, so $\frac{4a^3}{2015} + \frac{6a^2}{2015^2} + \frac{4a}{2015^3} + \frac{1}{2015^4} < \frac{4 \cdot 7^3 + 3}{2015} < 1$, so $a \geq 8$. When $a = 8$, we have $\frac{4a^3}{2015} = \frac{2048}{2015} > 1$, so $a = 8$ is the minimum.

Hence, the minimum n is $8^4 + 1 = \boxed{4097}$.

24. [**12**] Three ants begin on three different vertices of a tetrahedron. Every second, they choose one of the three edges connecting to the vertex they are on with equal probability and travel to the other vertex on that edge. They all stop when any two ants reach the same vertex at the same time. What is the probability that all three ants are at the same vertex when they stop?

Proposed by: Anna Ellison

Answer: $\boxed{\frac{1}{16}}$

At every second, each ant can travel to any of the three vertices they are not currently on. Given that, at one second, the three ants are on different vertices, the probability of them all going to the same vertex is $\frac{1}{27}$ and the probability of them all going to different vertices is $\frac{11}{27}$, so the probability of the three ants all meeting for the first time on the n^{th} step is $(\frac{11}{27})^{n-1} \times \frac{1}{27}$. Then the probability the three ants all meet at the same time is $\sum_{i=0}^{\infty} (\frac{11}{27})^i \times \frac{1}{27} = \frac{\frac{1}{27}}{1 - \frac{11}{27}} = \frac{1}{16}$.

25. [**13**] Let ABC be a triangle that satisfies $AB = 13, BC = 14, AC = 15$. Given a point P in the plane, let P_A, P_B, P_C be the reflections of A, B, C across P. Call P *good* if the circumcircle of $P_A P_B P_C$ intersects the circumcircle of ABC at exactly 1 point. The locus of good points P encloses a region $\mathcal{S}$. Find the area of $\mathcal{S}$.

Proposed by: Yang Liu

Answer: $\boxed{\frac{4225}{64}\pi}$

By the properties of reflection, the circumradius of $P_A P_B P_C$ equals the circumradius of ABC. Therefore, the circumcircle of $P_A P_B P_C$ must be externally tangent to the circumcircle of ABC. Now it's easy to see that the midpoint of the 2 centers of ABC and $P_A P_B P_C$ lies on the circumcircle of ABC. So the locus of P is simply the circumcircle of ABC.

Since $[ABC] = \frac{abc}{4R}$, we find the circumradius is $R = \frac{13 \cdot 14 \cdot 15}{84 \cdot 4} = \frac{65}{8}$, so the enclosed region has area $\frac{4225}{64}\pi$.

26. [**13**] Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a *continuous* function satisfying $f(xy) = f(x) + f(y) + 1$ for all positive reals x, y. If $f(2) = 0$, compute $f(2015)$.

Proposed by: Alexander Katz

Answer: $\boxed{\log_2 2015 - 1}$

Let $g(x) = f(x) + 1$. Substituting g into the functional equation, we get that

$$g(xy) - 1 = g(x) - 1 + g(y) - 1 + 1$$
$$g(xy) = g(x) + g(y).$$

Also, $g(2) = 1$. Now substitute $x = e^{x'}, y = e^{y'}$, which is possible because $x, y \in \mathbb{R}^+$. Then set $h(x) = g(e^x)$. This gives us that

$$g(e^{x'+y'}) = g(e^{x'}) + g(e^{y'}) \implies h(x' + y') = h(x') + h(y')$$

for al $x', y' \in \mathbb{R}$. Also h is continuous. Therefore, by Cauchy's functional equation, $h(x) = cx$ for a real number c. Going all the way back to g, we can get that $g(x) = c \log x$. Since $g(2) = 1, c = \frac{1}{\log 2}$. Therefore, $g(2015) = c \log 2015 = \frac{\log 2015}{\log 2} = \log_2 2015$.

Finally, $f(2015) = g(2015) - 1 = \log_2 2015 - 1$.

27. [**13**] Let $ABCD$ be a quadrilateral with $A = (3, 4), B = (9, -40), C = (-5, -12), D = (-7, 24)$. Let P be a point in the plane (not necessarily inside the quadrilateral). Find the minimum possible value of $AP + BP + CP + DP$.

Proposed by: Alexander Katz

Answer: $\boxed{16\sqrt{17} + 8\sqrt{5}}$

By the triangle inequality, $AP + CP \geq AC$ and $BP + DP \geq BD$. So P should be on AC and BD; i.e. it should be the intersection of the two diagonals. Then $AP + BP + CP + DP = AC + BD$, which is easily computed to be $16\sqrt{17} + 8\sqrt{5}$ by the Pythagorean theorem.

Note that we require the intersection of the diagonals to actually *exist* for this proof to work, but $ABCD$ is convex and this is not an issue.

28. [**15**] Find the shortest distance between the lines $\frac{x+2}{2} = \frac{y-1}{3} = \frac{z}{1}$ and $\frac{x-3}{-1} = \frac{y}{1} = \frac{z+1}{2}$

Proposed by: Sam Korsky

Answer: $\boxed{\frac{5\sqrt{3}}{3}}$

First we find the direction of a line perpendicular to both of these lines. By taking the cross product $(2, 3, 1) \times (-1, 1, 2) = (5, -5, 5)$ we find that the plane $x - y + z + 3 = 0$ contains the first line and is parallel to the second. Now we take a point on the second line, say the point $(3, 0, -1)$ and find the distance between this point and the plane. This comes out to $\frac{|3-0+(-1)+3|}{\sqrt{1^2+1^2+1^2}} = \frac{5}{\sqrt{3}} = \boxed{\frac{5\sqrt{3}}{3}}$.

29. [**15**] Find the largest real number k such that there exists a sequence of positive reals $\{a_i\}$ for which $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n^k}$ does not.

Proposed by: Alexander Katz

Answer: $\boxed{\frac{1}{2}}$

For $k > \frac{1}{2}$, I claim that the second sequence must converge. The proof is as follows: by the Cauchy-Schwarz inequality,

$$\left(\sum_{n \geq 1} \frac{\sqrt{a_n}}{n^k}\right)^2 \leq \left(\sum_{n \geq 1} a_n\right) \left(\sum_{n \geq 1} \frac{1}{n^{2k}}\right)$$

Since for $k > \frac{1}{2}, \sum_{n \geq 1} \frac{1}{n^{2k}}$ converges, the right hand side converges. Therefore, the left hand side must also converge.

For $k \leq \frac{1}{2}$, the following construction surprisingly works: $a_n = \frac{1}{n \log^2 n}$. It can be easily verified that $\sum_{n\geq 1} a_n$ converges, while

$$\sum_{n\geq 1} \frac{\sqrt{a_n}}{n^{\frac{1}{2}}} = \sum_{n\geq 1} \frac{1}{n \log n}$$

does not converge.

30. [**15**] Find the largest integer n such that the following holds: there exists a set of n points in the plane such that, for any choice of three of them, some two are unit distance apart.

Proposed by: Alexander Katz

Answer: $\boxed{7}$

We can obtain $n = 7$ in the following way: Consider a rhombus $ABCD$ made up of two equilateral triangles of side length 1, where $\angle DAB = 60°$. Rotate the rhombus clockwise about A to obtain a new rhombus $AB'C'D'$ such that $DD' = 1$. Then one can verify that the seven points A, B, C, D, B', C', D' satisfy the problem condition.

To prove that $n = 8$ points is unobtainable, one interprets the problem in terms of graph theory. Consider a graph on 8 vertices, with an edge drawn between two vertices if and only if the vertices are at distance 1 apart. Assume for the sake of contradiction that this graph has no three points, no two of which are at distance 1 apart (in terms of graph theory, this means the graph has no independent set of size 3).

First, note that this graph cannot contain a complete graph of size 4 (it's clear that there can't exist four points in the plane with any two having the same pairwise distance).

I claim that every vertex has degree 4. It is easy to see that if a vertex has degree 5 or higher, then there exists an independent set of size 3 among its neighbors, contradiction (one can see this by drawing the 5 neighbors on a circle of radius 1 centered at our initial vertex and considering their pairwise distances). Moreover, if a vertex has degree 3 or lower then there are at least four vertices that are not at distance 1 from that vertex, and since not all four of these vertices can be at distance 1 from one another, there exists an independent set of of size 3, contradiction.

Now, we consider the complement of our graph. Every vertex of this new graph has degree 3 and by our observations, contains no independent set of size 4. Moreover, by assumption this graph contains no triangle (a complete graph on three vertices). But we can check by hand that there are only six distinct graphs on eight vertices with each vertex having degree 3 (up to isomorphism), and five of these graphs contain a triangle, and the remaining graph contains an independent set of size 4, contradiction!

Hence the answer is $\boxed{n = 7}$

31. [**17**] Two random points are chosen on a segment and the segment is divided at each of these two points. Of the three segments obtained, find the probability that the largest segment is more than three times longer than the smallest segment.

Proposed by: Sam Korsky

Answer: $\boxed{\frac{27}{35}}$

We interpret the problem with geometric probability. Let the three segments have lengths $x, y, 1-x-y$ and assume WLOG that $x \geq y \geq 1-x-y$. The every possible (x, y) can be found in the triangle determined by the points $\left(\frac{1}{3}, \frac{1}{3}\right)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$, $(1, 0)$ in $\mathbb{R}^2$, which has area $\frac{1}{12}$. The line $x = 3(1-x-y)$ intersects the lines $x = y$ and $y = 1-x-y$ at the points $\left(\frac{3}{7}, \frac{3}{7}\right)$ and $\left(\frac{3}{5}, \frac{1}{5}\right)$ Hence $x \leq 3(1-x-y)$ if (x, y) is in the triangle determined by points $\left(\frac{1}{3}, \frac{1}{3}\right)$, $\left(\frac{3}{7}, \frac{3}{7}\right)$, $\left(\frac{3}{5}, \frac{1}{5}\right)$ which by shoelace has area $\frac{2}{105}$. Hence the desired probability is given by

$$\frac{\frac{1}{12} - \frac{2}{105}}{\frac{1}{12}} = \boxed{\frac{27}{35}}$$

32. [**17**] Find the sum of all positive integers $n \leq 2015$ that can be expressed in the form $\lceil \frac{x}{2} \rceil + y + xy$, where x and y are positive integers.

Proposed by: Calvin Deng

Answer: $\boxed{2029906}$

Lemma: n is expressible as $\lceil \frac{x}{2} \rceil + y + xy$ iff $2n + 1$ is not a Fermat Prime.

Proof: Suppose n is expressible. If $x = 2k$, then $2n + 1 = (2k + 1)(2y + 1)$, and if $x = 2k - 1$, then $n = k(2y + 1)$. Thus, if $2n + 1$ isn't prime, we can factor $2n + 1$ as the product of two odd integers $2x + 1$, $2y + 1$ both greater than 1, resulting in positive integer values for x and y. Also, if n has an odd factor greater than 1, then we factor out its largest odd factor as $2y + 1$, giving a positive integer value for x and y. Thus n is expressible iff $2n + 1$ is not prime or n is not a power of 2. That leaves only the n such that $2n + 1$ is a prime one more than a power of two. These are well-known, and are called the Fermat primes.

It's a well-known fact that the only Fermat primes ≤ 2015 are 3, 5, 17, 257, which correspond to $n = 1, 2, 8, 128$. Thus the sum of all expressible numbers is $\frac{2015 \cdot 2016}{2} - (1 + 2 + 8 + 128) = 2029906$.

33. [**17**] How many ways are there to place four points in the plane such that the set of pairwise distances between the points consists of exactly 2 elements? (Two configurations are the same if one can be obtained from the other via rotation and scaling.)

Proposed by: Alexander Katz

Answer: $\boxed{6}$

Let A, B, C, D be the four points. There are 6 pairwise distances, so at least three of them must be equal.

Case 1: There is no equilateral triangle. Then WLOG we have $AB = BC = CD = 1$.

- Subcase 1.1: $AD = 1$ as well. Then $AC = BD \neq 1$, so $ABCD$ is a square.
- Subcase 1.2: $AD \neq 1$. Then $AC = BD = AD$, so A, B, C, D are four points of a regular pentagon.

Case 2: There is an equilateral triangle, say ABC, of side length 1.

- Subcase 2.1: There are no more pairs of distance 1. Then D must be the center of the triangle.
- Subcase 2.2: There is one more pair of distance 1, say AD. Then D can be either of the two intersections of the unit circle centered at A with the perpendicular bisector of BC. This gives us 2 kites.
- Subcase 2.3: Both $AD = BD = 1$. Then $ABCD$ is a rhombus with a $60°$ angle.

This gives us 6 configurations total.

34. [**20**] Let n be the **second** smallest integer that can be written as the sum of two positive cubes in two different ways. Compute n. If your guess is a, you will receive $\max\left(25 - 5 \cdot \max\left(\frac{a}{n}, \frac{n}{a}\right), 0\right)$ points, rounded up.

Proposed by: Alexander Katz

Answer: $\boxed{4104}$

A computer search yields that the second smallest number is 4104. Indeed, $4104 = 9^3 + 15^3 = 2^3 + 16^3$

35. [**20**] Let n be the smallest positive integer such that any positive integer can be expressed as the sum of n integer 2015th powers. Find n. If your answer is a, your score will be $\max(20 - \frac{1}{5}|\log_{10} \frac{a}{n}|, 0)$, rounded up.

Proposed by: Alexander Katz

Answer: $\boxed{2^{2015} + \left\lfloor \left(\frac{3}{2}\right)^{2015} \right\rfloor - 2}$

In general, if $k \le 471600000$, then any integer can be expressed as the sum of $2^k + \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor - 2$ integer kth powers. This bound is optimal.

The problem asking for the minimum number of k-th powers needed to add to any positive integer is called Waring's problem.

36. [**20**] Consider the following seven false conjectures with absurdly high counterexamples. Pick any subset of them, and list their labels in order of their smallest counterexample (the smallest n for which the conjecture is false) from smallest to largest. For example, if you believe that the below list is already ordered by counterexample size, you should write "PECRSGA".

- P. (**Polya's conjecture**) For any integer n, at least half of the natural numbers below n have an odd number of prime factors.
- E. (**Euler's conjecture**) There is no perfect cube n that can be written as the sum of three positive cubes.
- C. (**Cyclotomic**) The polynomial with minimal degree whose roots are the primitive nth roots of unity has all coefficients equal to -1, 0, or 1.
- R. (**Prime race**) For any integer n, there are more primes below n equal to $2 \pmod 3$ than there are equal to $1 \pmod 3$.
- S. (**Seventeen conjecture**) For any integer n, $n^{17} + 9$ and $(n+1)^{17} + 9$ are relatively prime.
- G. (**Goldbach's (other) conjecture**) Any odd composite integer n can be written as the sum of a prime and twice a square.
- A. (**Average square**) Let $a_1 = 1$ and $a_{k+1} = \frac{1+a_1^2+a_2^2+\ldots+a_k^2}{k}$. Then a_n is an integer for any n.

If your answer is a list of $4 \le n \le 7$ labels in the correct order, your score will be $(n-2)(n-3)$. Otherwise, it will be 0.

Proposed by: Alexander Katz

Answer: ACGPRES

The smallest counterexamples are:

- Polya's conjecture: 906,150,257
- Euler's sum of powers: 31,858,749,840,007,945,920,321
- Cyclotomic polynomials: 105
- Prime race: 23,338,590,792
- Seventeen conjecture: 8,424,432,925,592,889,329,288,197,322,308,900,672,459,420,460,792,433
- Goldbach's other conjecture: 5777
- Average square: 44