HMMT November 2015

November 14, 2015

Team

1. [**3**] Triangle ABC is isosceles, and $\angle ABC = x^\circ$. If the sum of the possible measures of $\angle BAC$ is 240°, find x.

2. [**3**] Bassanio has three red coins, four yellow coins, and five blue coins. At any point, he may give Shylock any two coins of different colors in exchange for one coin of the other color; for example, he may give Shylock one red coin and one blue coin, and receive one yellow coin in return. Bassanio wishes to end with coins that are all the same color, and he wishes to do this while having as many coins as possible. How many coins will he end up with, and what color will they be?

3. [**3**] Let $\lfloor x \rfloor$ denote the largest integer less than or equal to x, and let $\{x\}$ denote the fractional part of x. For example, $\lfloor \pi \rfloor = 3$, and $\{\pi\} = 0.14159\ldots$, while $\lfloor 100 \rfloor = 100$ and $\{100\} = 0$. If n is the largest solution to the equation $\frac{\lfloor n \rfloor}{n} = \frac{2015}{2016}$, compute $\{n\}$.

4. [**5**] Call a set of positive integers *good* if there is a partition of it into two sets S and T, such that there do not exist three elements $a, b, c \in S$ such that $a^b = c$ and such that there do not exist three elements $a, b, c \in T$ such that $a^b = c$ (a and b need not be distinct). Find the smallest positive integer n such that the set $\{2, 3, 4, \ldots, n\}$ is *not* good.

5. [**5**] Kelvin the Frog is trying to hop across a river. The river has 10 lilypads on it, and he must hop on them in a specific order (the order is unknown to Kelvin). If Kelvin hops to the wrong lilypad at any point, he will be thrown back to the wrong side of the river and will have to start over. Assuming Kelvin is infinitely intelligent, what is the minimum number of hops he will need to guarantee reaching the other side?

6. [**5**] Marcus and four of his relatives are at a party. Each pair of the five people are either *friends* or *enemies*. For any two enemies, there is no person that they are both friends with. In how many ways is this possible?

7. [**6**] Let $ABCD$ be a convex quadrilateral whose diagonals AC and BD meet at P. Let the area of triangle APB be 24 and let the area of triangle CPD be 25. What is the minimum possible area of quadrilateral $ABCD$?

8. [**6**] Find **any** quadruple of positive integers (a, b, c, d) satisfying $a^3 + b^4 + c^5 = d^{11}$ and $abc < 10^5$.

9. [**7**] A graph consists of 6 vertices. For each pair of vertices, a coin is flipped, and an edge connecting the two vertices is drawn if and only if the coin shows heads. Such a graph is *good* if, starting from any vertex V connected to at least one other vertex, it is possible to draw a path starting and ending at V that traverses each edge exactly once. What is the probability that the graph is good?

10. [**7**] A number n is *bad* if there exists some integer c for which $x^x \equiv c \pmod{n}$ has no integer solutions for x. Find the number of bad integers between 2 and 42 inclusive.