HMMT November 2015

November 14, 2015

Team

1. [**3**] Triangle ABC is isosceles, and $\angle ABC = x^\circ$. If the sum of the possible measures of $\angle BAC$ is 240°, find x.

   Proposed by: Alexander Katz

   Answer: $\boxed{20}$

   There are three possible triangles: either $\angle ABC = \angle BCA$, in which case $\angle BAC = 180 - 2x$, $\angle ABC = \angle BAC$, in which case $\angle BAC = x$, or $\angle BAC = \angle BCA$, in which case $\angle BAC = \frac{180-x}{2}$. These sum to $\frac{540-3x}{2}$, so we have $\frac{540-3x}{2} = 240 \implies x = \boxed{20}$.

2. [**3**] Bassanio has three red coins, four yellow coins, and five blue coins. At any point, he may give Shylock any two coins of different colors in exchange for one coin of the other color; for example, he may give Shylock one red coin and one blue coin, and receive one yellow coin in return. Bassanio wishes to end with coins that are all the same color, and he wishes to do this while having as many coins as possible. How many coins will he end up with, and what color will they be?

   Proposed by: Yang Liu

   Answer: $\boxed{\text{7 yellow coins}}$

   Let r, y, b denote the numbers of red, yellow, and blue coins respectively. Note that each of the three possible exchanges do not change the parities of $y - r, b - y$, or $b - r$, and eventually one of these differences becomes zero. Since $b - r$ is the only one of these differences that is originally even, it must be the one that becomes zero, and so Bassanio will end with some number of yellow coins. Furthermore, Bassanio loses a coin in each exchange, and he requires at least five exchanges to rid himself of the blue coins, so he will have at most $12 - 5 = 7$ yellow coins at the end of his trading.

   It remains to construct a sequence of trades that result in seven yellow coins. First, Bassanio will exchange one yellow and one blue coin for one red coin, leaving him with four red coins, three yellow coins, and four blue coins. He then converts the red and blue coins into yellow coins, resulting in $\boxed{\text{7 yellow coins}}$, as desired.

3. [**3**] Let $\lfloor x \rfloor$ denote the largest integer less than or equal to x, and let $\{x\}$ denote the fractional part of x. For example, $\lfloor \pi \rfloor = 3$, and $\{\pi\} = 0.14159\ldots$, while $\lfloor 100 \rfloor = 100$ and $\{100\} = 0$. If n is the largest solution to the equation $\frac{\lfloor n \rfloor}{n} = \frac{2015}{2016}$, compute $\{n\}$.

   Proposed by: Alexander Katz

   Answer: $\boxed{\frac{2014}{2015}}$

   Note that $n = \lfloor n \rfloor + \{n\}$, so

$$
\begin{aligned}
\frac{\lfloor n \rfloor}{n} &= \frac{\lfloor n \rfloor}{\lfloor n \rfloor + \{n\}} \\
&= \frac{2015}{2016} \\
\implies 2016\lfloor n \rfloor &= 2015\lfloor n \rfloor + 2015\{n\} \\
\implies \lfloor n \rfloor &= 2015\{n\}
\end{aligned}
$$

   Hence, $n = \lfloor n \rfloor + \{n\} = \frac{2016}{2015}\lfloor n \rfloor$, and so n is maximized when $\lfloor n \rfloor$ is also maximized. As $\lfloor n \rfloor$ is an integer, and $\{n\} < 1$, the maximum possible value of $\lfloor n \rfloor$ is 2014. Therefore, $\{n\} = \frac{\lfloor n \rfloor}{2015} = \boxed{\frac{2014}{2015}}$.

4. [**5**] Call a set of positive integers *good* if there is a partition of it into two sets S and T, such that there do not exist three elements $a, b, c \in S$ such that $a^b = c$ and such that there do not exist three elements $a, b, c \in T$ such that $a^b = c$ (a and b need not be distinct). Find the smallest positive integer n such that the set $\{2, 3, 4, \ldots, n\}$ is *not* good.

   Proposed by: Sam Korsky

   Answer: $\boxed{65536}$

   First, we claim that the set $\{2, 4, 8, 256, 65536\}$ is not good. Assume the contrary and say $2 \in S$. Then since $2^2 = 4$, we have $4 \in T$. And since $4^4 = 256$, we have $256 \in S$. Then since $256^2 = 65536$, we have $65536 \in T$. Now, note that we cannot place 8 in either S or T, contradiction.

   Hence $n \leq 65536$. And the partition $S = \{2, 3\} \cup \{256, 257, \ldots, 65535\}$ and $T = \{4, 5, \ldots, 255\}$ shows that $n \geq 65536$. Therefore $n = \boxed{65536}$.

5. [**5**] Kelvin the Frog is trying to hop across a river. The river has 10 lilypads on it, and he must hop on them in a specific order (the order is unknown to Kelvin). If Kelvin hops to the wrong lilypad at any point, he will be thrown back to the wrong side of the river and will have to start over. Assuming Kelvin is infinitely intelligent, what is the minimum number of hops he will need to guarantee reaching the other side?

   Proposed by: Alexander Katz

   Answer: $\boxed{176}$

   Kelvin needs (at most) $i(10-i)$ hops to determine the ith lilypad he should jump to, then an additional 11 hops to actually get across the river. Thus he requires $\sum_{i=1}^{10} i(10-i) + 11 = \boxed{176}$ hops to guarantee success.

6. [**5**] Marcus and four of his relatives are at a party. Each pair of the five people are either *friends* or *enemies*. For any two enemies, there is no person that they are both friends with. In how many ways is this possible?

   Proposed by: Alexander Katz

   Answer: $\boxed{52}$

   Denote friendship between two people a and b by $a \sim b$. Then, assuming everyone is friends with themselves, the following conditions are satisfied:

   - $a \sim a$
   - If $a \sim b$, then $b \sim a$
   - If $a \sim b$ and $b \sim c$, then $a \sim c$

   Thus we can separate the five people into a few groups (possibly one group), such that people are friends within each group, but two people are enemies when they are in different groups. Here comes the calculation. Since the number of group(s) can be 1, 2, 3, 4, or 5, we calculate for each of those cases. When there's only one group, then we only have 1 possibility that we have a group of 5, and the total number of friendship assignments in this case is $\binom{5}{5} = 1$; when there are two groups, we have $5 = 1 + 4 = 2 + 3$ are all possible numbers of the two groups, with a total of $\binom{5}{1} + \binom{5}{2} = 15$ choices; when there are three groups, then we have $5 = 1 + 1 + 3 = 1 + 2 + 2$, with $\binom{5}{3} + \frac{\binom{5}{1}\binom{5}{2}}{2} = 25$ possibilities; when there are four of them, then we have $5 = 1 + 1 + 1 + 2$ be its only possibility, with $\binom{5}{2} = 10$ separations; when there are 5 groups, obviously we have 1 possibility. Hence, we have a total of $1 + 15 + 25 + 10 + 1 = \boxed{52}$ possibilities.

   Alternatively, we can also solve the problem recursively. Let B_n be the number of friendship graphs with n people, and consider an arbitrary group. If this group has size k, then there are $\binom{n}{k}$ possible such groups, and B_{n-k} friendship graphs on the remaining $n - k$ people. Therefore, we have the recursion

$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}$$

with the initial condition $B_1 = 1$. Calculating routinely gives $B_5 = 52$ as before.

7. [**6**] Let $ABCD$ be a convex quadrilateral whose diagonals AC and BD meet at P. Let the area of triangle APB be 24 and let the area of triangle CPD be 25. What is the minimum possible area of quadrilateral $ABCD$?

   Proposed by: Sam Korsky

   Answer: $\boxed{49 + 20\sqrt{6}}$

   Note that $\angle APB = 180^\circ - \angle BPC = \angle CPD = 180^\circ - \angle DPA$ so $4[BPC][DPA] = (PB \cdot PC \cdot \sin BPC)(PD \cdot PA \cdot \sin DPA) = (PA \cdot PB \cdot \sin APB)(PC \cdot PD \cdot \sin CPD) = 4[APB][CPD] = 2400 \implies [BPC][DPA] = 600$. Hence by AM-GM we have that

   $$[BPC] + [DPA] \geq 2\sqrt{[BPC][DPA]} = 20\sqrt{6}$$

   so the minimum area of quadrilateral $ABCD$ is $\boxed{49 + 20\sqrt{6}}$.

8. [**6**] Find **any** quadruple of positive integers (a, b, c, d) satisfying $a^3 + b^4 + c^5 = d^{11}$ and $abc < 10^5$.

   Proposed by: Sam Korsky

   Answer: $\boxed{(a, b, c, d) = (128, 32, 16, 4) \text{ or } (a, b, c, d) = (160, 16, 8, 4)}$

   It's easy to guess that there are solutions such that a, b, c, d are in the form of n^x, where n is a rather small number. After a few attempts, we can see that we obtain simple equations when $n = 2$ or $n = 3$: for $n = 2$, the equation becomes in the form of $2^t + 2^t + 2^{t+1} = 2^{t+2}$ for some non-negative integer t; for $n = 3$, the equation becomes in the form of $3^t + 3^t + 3^t = 3^{t+1}$ for some non-negative integer t. In the first case, we hope that t is a multiple of two of $3, 4, 5$, that $t + 1$ is a multiple of the last one, and that $t + 2$ is a multiple of 11. Therefore, $t \equiv 15, 20, 24 \pmod{60}$ and $t \equiv 9 \pmod{11}$. It's easy to check that the only solution that satisfies the given inequality is the solution with $t = 20$, and $(a, b, c, d) = (128, 32, 16, 4)$. In the case where $n = 3$, we must have that t is a multiple of 60, which obviously doesn't satisfy the inequality restriction. Remark: By programming, we find that the only two solutions are $(a, b, c, d) = (128, 32, 16, 4)$ and $(a, b, c, d) = (160, 16, 8, 4)$, with the the former being the intended solution.

9. [**7**] A graph consists of 6 vertices. For each pair of vertices, a coin is flipped, and an edge connecting the two vertices is drawn if and only if the coin shows heads. Such a graph is *good* if, starting from any vertex V connected to at least one other vertex, it is possible to draw a path starting and ending at V that traverses each edge exactly once. What is the probability that the graph is good?

   Proposed by: Sam Korsky

   Answer: $\boxed{\frac{507}{16384} \text{ or } \frac{2^{10}-10}{2^{15}} \text{ or } \frac{2^9-5}{2^{14}}}$

   First, we find the probability that all vertices have even degree. Arbitrarily number the vertices 1, 2, 3, 4, 5, 6. Flip the coin for all the edges out of vertex 1; this vertex ends up with even degree with probability $\frac{1}{2}$. Next we flip for all the remaining edges out of vertex 2; regardless of previous edges, vertex 2 ends up with even degree with probability $\frac{1}{2}$, and so on through vertex 5. Finally, if vertices 1 through 5 all have even degree, vertex 6 must also have even degree. So all vertices have even degree with probability $\frac{1}{2^5} = \frac{1}{32}$. There are $\binom{6}{2} = 15$ edges total, so there are 2^{15} total possible graphs, of which 2^{10} have all vertices with even degree. Observe that exactly 10 of these latter graphs are not good, namely, the $\frac{1}{2}\binom{6}{3}$ graphs composed of two separate triangles. So $2^{10} - 10$ of our graphs are good, and the probability that a graph is good is $\frac{2^{10}-10}{2^{15}}$.

10. [**7**] A number n is *bad* if there exists some integer c for which $x^x \equiv c \pmod{n}$ has no integer solutions for x. Find the number of bad integers between 2 and 42 inclusive.

    Proposed by: Sam Korsky

    Answer: $\boxed{25}$

Call a number *good* if it is not *bad*. We claim all good numbers are products of distinct primes, none of which are equivalent to 1 modulo another.

We first show that all such numbers are *good*. Consider $n = p_1 p_2 \dots p_k$, and let x be a number satisfying $x \equiv c \pmod{p_1 p_2 \dots p_k}$ and $x \equiv 1 \pmod{(p_1 - 1)(p_2 - 1) \dots (p_k - 1)}$. Since, by assumption, $p_1 p_2 \dots p_k$ and $(p_1 - 1)(p_2 - 1) \dots (p_k - 1)$ are relatively prime, such an x must exist by CRT. Then $x^x \equiv c^1 = c \pmod{n}$, for any c, as desired.

We now show that all other numbers are *bad*. Suppose that there exist some $p_1, p_2 \mid n$ such that $\gcd(p_1, p_2 - 1) \neq 1$ (which must hold for some two primes by assumption), and hence $\gcd(p_1, p_2 - 1) = p_1$. Consider some c for which $p_1 c$ is not a p_1th power modulo p_2, which must exist as $p_1 c$ can take any value modulo p_2 (as p_1, p_2 are relatively prime). We then claim that $x^x \equiv p_1 c \pmod{n}$ is not solvable.

Since $p_1 p_2 \mid n$, we have $x^x \equiv p_1 c \pmod{p_1 p_2}$, hence $p_1 \mid x$. But then $x^x \equiv p_1 c$ is a p_1th power modulo p_2 as $p_1 \mid x$, contradicting our choice of c. As a result, all such numbers are *bad*.

Finally, it is easy to see that n is *bad* if it is not squarefree. If p_1 divides n twice, then letting $c = p_1$ makes the given equivalence unsolvable.

Hence, there are 16 numbers (13 primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41; and 3 semiprimes: $3 \cdot 5 = 15$, $3 \cdot 11 = 33$, $5 \cdot 7 = 35$) that are *good*, which means that $41 - 16 = \boxed{25}$ numbers are *bad*.