HMMT November 2015

November 14, 2015

Theme round

1. Consider a 1×1 grid of squares. Let A, B, C, D be the vertices of this square, and let E be the midpoint of segment CD. Furthermore, let F be the point on segment BC satisfying $BF = 2CF$, and let P be the intersection of lines AF and BE. Find $\frac{AP}{PF}$.

2. Consider a 2×2 grid of squares. David writes a positive integer in each of the squares. Next to each row, he writes the product of the numbers in the row, and next to each column, he writes the product of the numbers in each column. If the sum of the eight numbers he writes down is 2015, what is the minimum possible sum of the four numbers he writes in the grid?

3. Consider a 3×3 grid of squares. A circle is inscribed in the lower left corner, the middle square of the top row, and the rightmost square of the middle row, and a circle O with radius r is drawn such that O is externally tangent to each of the three inscribed circles. If the side length of each square is 1, compute r.

4. Consider a 4×4 grid of squares. Aziraphale and Crowley play a game on this grid, alternating turns, with Aziraphale going first. On Aziraphales turn, he may color any uncolored square red, and on Crowleys turn, he may color any uncolored square blue. The game ends when all the squares are colored, and Aziraphales score is the area of the largest closed region that is entirely red. If Aziraphale wishes to maximize his score, Crowley wishes to minimize it, and both players play optimally, what will Aziraphales score be?

5. Consider a 5×5 grid of squares. Vladimir colors some of these squares red, such that the centers of any four red squares do **not** form an axis-parallel rectangle (i.e. a rectangle whose sides are parallel to those of the squares). What is the maximum number of squares he could have colored red?

6. Consider a 6×6 grid of squares. Edmond chooses four of these squares uniformly at random. What is the probability that the centers of these four squares form a square?

7. Consider a 7×7 grid of squares. Let $f : \{1, 2, 3, 4, 5, 6, 7\} \to \{1, 2, 3, 4, 5, 6, 7\}$ be a function; in other words, $f(1), f(2), \ldots, f(7)$ are each (not necessarily distinct) integers from 1 to 7. In the top row of the grid, the numbers from 1 to 7 are written in order; in every other square, $f(x)$ is written where x is the number above the square. How many functions have the property that the bottom row is identical to the top row, and no other row is identical to the top row?

8. Consider an 8×8 grid of squares. A rook is placed in the lower left corner, and every minute it moves to a square in the same row or column with equal probability (the rook must move; i.e. it cannot stay in the same square). What is the expected number of minutes until the rook reaches the upper right corner?

9. Consider a 9×9 grid of squares. Haruki fills each square in this grid with an integer between 1 and 9, inclusive. The grid is called a *super-sudoku* if each of the following three conditions hold:

   - Each column in the grid contains each of the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ exactly once.
   - Each row in the grid contains each of the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ exactly once.
   - Each 3×3 subsquare in the grid contains each of the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ exactly once.

   How many possible super-sudoku grids are there?

10. Consider a 10×10 grid of squares. One day, Daniel drops a burrito in the top left square, where a wingless pigeon happens to be looking for food. Every minute, if the pigeon and the burrito are in the same square, the pigeon will eat 10% of the burrito's original size and accidentally throw it into a random square (possibly the one it is already in). Otherwise, the pigeon will move to an adjacent square, decreasing the distance between it and the burrito. What is the expected number of minutes before the pigeon has eaten the entire burrito?