HMMT November 2015

November 14, 2015

Theme round

1. Consider a 1×1 grid of squares. Let A, B, C, D be the vertices of this square, and let E be the midpoint of segment CD. Furthermore, let F be the point on segment BC satisfying $BF = 2CF$, and let P be the intersection of lines AF and BE. Find $\frac{AP}{PF}$.

   Proposed by: Sam Korsky

   Answer: $\boxed{3}$

   Let line BE hit line DA at Q. It's clear that triangles AQP and FBP are similar so

$$\frac{AP}{PF} = \frac{AQ}{BF} = \frac{2AD}{\frac{2}{3}BC} = \boxed{3}$$

2. Consider a 2×2 grid of squares. David writes a positive integer in each of the squares. Next to each row, he writes the product of the numbers in the row, and next to each column, he writes the product of the numbers in each column. If the sum of the eight numbers he writes down is 2015, what is the minimum possible sum of the four numbers he writes in the grid?

   Proposed by: Alexander Katz

   Answer: $\boxed{88}$

   Let the four numbers be a, b, c, d, so that the other four numbers are ab, ad, bc, bd. The sum of these eight numbers is $a + b + c + d + ab + ad + bc + bd = (a + c) + (b + d) + (a + c)(b + d) = 2015$, and so $(a + c + 1)(b + d + 1) = 2016$. Since we seek to minimize $a + b + c + d$, we need to find the two factors of 2016 that are closest to each other, which is easily calculated to be $42 \cdot 48 = 2016$; this makes $a + b + c + d = \boxed{88}$.

3. Consider a 3×3 grid of squares. A circle is inscribed in the lower left corner, the middle square of the top row, and the rightmost square of the middle row, and a circle O with radius r is drawn such that O is externally tangent to each of the three inscribed circles. If the side length of each square is 1, compute r.

   Proposed by: Sam Korsky

   Answer: $\boxed{\frac{5\sqrt{2}-3}{6}}$

   Let A be the center of the square in the lower left corner, let B be the center of the square in the middle of the top row, and let C be the center of the rightmost square in the middle row. It's clear that O is the circumcenter of triangle ABC - hence, the desired radius is merely the circumradius of triangle ABC minus $\frac{1}{2}$. Now note that by the Pythagorean theorem, $BC = \sqrt{2}$ and $AB = AC = \sqrt{5}$ so we easily find that the altitude from A in triangle ABC has length $\frac{3\sqrt{2}}{2}$. Therefore the area of triangle ABC is $\frac{3}{2}$. Hence the circumradius of triangle ABC is given by

$$\frac{BC \cdot CA \cdot AB}{4 \cdot \frac{3}{2}} = \frac{5\sqrt{2}}{6}$$

   and so the answer is $\frac{5\sqrt{2}}{6} - \frac{1}{2} = \boxed{\frac{5\sqrt{2}-3}{6}}$.

4. Consider a 4×4 grid of squares. Aziraphale and Crowley play a game on this grid, alternating turns, with Aziraphale going first. On Aziraphales turn, he may color any uncolored square red, and on Crowleys turn, he may color any uncolored square blue. The game ends when all the squares are colored, and Aziraphales score is the area of the largest closed region that is entirely red. If Aziraphale wishes to maximize his score, Crowley wishes to minimize it, and both players play optimally, what will Aziraphales score be?

Proposed by: Alexander Katz

Answer: $\boxed{6}$

We claim that the answer is 6.

On Aziraphale's first two turns, it is always possible for him to take 2 adjacent squares from the central four; without loss of generality, suppose they are the squares at $(1, 1)$ and $(1, 2)$. If allowed, Aziraphale's next turn will be to take one of the remaining squares in the center, at which point there will be seven squares adjacent to a red square, and so Aziraphale can guarantee at least two more adjacent red squares. After that, since the number of blue squares is always at most the number of red squares, Aziraphale can guarantee another adjacent red square, making his score at least 6.

If, however, Crowley does not allow Aziraphale to attain another central red square – i.e. coloring the other two central squares blue – then Aziraphale will continue to take squares from the second row, WLOG $(1, 3)$. If Aziraphale is also allowed to take $(1, 0)$, he will clearly attain at least 6 adjacent red squares as each red square in this row has two adjacent squares to it, and otherwise (if Crowley takes $(1, 0)$), Aziraphale will take $(0, 1)$ and guarantee a score of at least $4 + \frac{4}{2} = 6$ as there are 4 uncolored squares adjacent to a red one.

Therefore, the end score will be at least 6. We now show that this is the best possible for Aziraphale; i.e. Crowley can always limit the score to 6. Crowley can play by the following strategy: if Aziraphale colors a square in the second row, Crowley will color the square below it, if Aziraphale colors a square in the third row, Crowley will color the square above it. Otherwise, if Aziraphale colors a square in the first or fourth rows, Crowley will color an arbitrary square in the same row. It is clear that the two "halves" of the board cannot be connected by red squares, and so the largest contiguous red region will occur entirely in one half of the grid, but then the maximum score is $4 + \frac{4}{2} = 6$.

The optimal score is thus both at least 6 and at most 6, so it must be 6 as desired.

5. Consider a 5×5 grid of squares. Vladimir colors some of these squares red, such that the centers of any four red squares do **not** form an axis-parallel rectangle (i.e. a rectangle whose sides are parallel to those of the squares). What is the maximum number of squares he could have colored red?

Proposed by: Sam Korsky

Answer: $\boxed{12}$

We claim that the answer is 12. We first show that if 13 squares are colored red, then some four form an axis-parallel rectangle. Note that we can swap both columns and rows without affecting whether four squares form a rectangle, so we may assume without loss of generality that the top row has the most red squares colored; suppose it has k squares colored. We may further suppose that, without loss of generality, these k red squares are the first k squares in the top row from the left.

Consider the $k \times 5$ rectangle formed by the first k columns. In this rectangle, no more than 1 square per row can be red (excluding the top one), so there are a maximum of $k+4$ squares colored red. In the remaining $(5-k)\times 5$ rectangle, at most $4(5-k)$ squares are colored red (as the top row of this rectangle has no red squares), so there are a maximum of $(k + 4) + 4(5 - k) = 24 - 3k$ squares colored red in the 5×5 grid. By assumption, at least 13 squares are colored red, so we have $13 \leq 24 - 3k \iff k \leq 3$.

Hence there are at most 3 red squares in any row. As there are at least 13 squares colored red, this implies that at least 3 rows have 3 red squares colored. Consider the 3×5 rectangle formed by these three rows. Suppose without loss of generality that the leftmost three squares in the top row are colored red, which forces the rightmost three squares in the second row to be colored red. But then, by the Pigeonhole Principle, some 2 of the 3 leftmost squares or some 2 of the 3 rightmost squares in the bottom row will be colored red, leading to an axis-parallel rectangle – a contradiction.

Hence there are most 12 squares colored red. It remains to show that there exists some coloring where exactly 12 squares are colored red, one example of which is illustrated below:

	R	R	R	R
R	R			
R		R		
R			R	
R				R

The maximum number of red squares, therefore, is $\boxed{12}$.

6. Consider a 6×6 grid of squares. Edmond chooses four of these squares uniformly at random. What is the probability that the centers of these four squares form a square?

   Proposed by: Alexander Katz

   Answer: $\boxed{\frac{1}{561} \text{ OR } \frac{105}{\binom{36}{4}}}$

   Firstly, there are $\binom{36}{4}$ possible combinations of points. Call a square *proper* if its sides are parallel to the coordinate axes and *improper* otherwise. Note that every *improper* square can be inscribed in a unique *proper* square. Hence, an $n \times n$ *proper* square represents a total of n squares: 1 proper and $n-1$ improper.

   There are thus a total of

$$\begin{aligned}\sum_{i=1}^{6} i(6-i)^2 &= \sum_{i=1}^{6}(i^3 - 12i^2 + 36i) \\ &= \sum_{i=1}^{6} i^3 - 12\sum_{i=1}^{6} i^2 + 36\sum i = 1^6 i \\ &= 441 - 12(91) + 36(21) \\ &= 441 - 1092 + 756 \\ &= 105\end{aligned}$$

   squares on the grid. Our desired probability is thus $\frac{105}{\binom{36}{4}} = \boxed{\frac{1}{561}}$.

7. Consider a 7×7 grid of squares. Let $f : \{1, 2, 3, 4, 5, 6, 7\} \to \{1, 2, 3, 4, 5, 6, 7\}$ be a function; in other words, $f(1), f(2), \ldots, f(7)$ are each (not necessarily distinct) integers from 1 to 7. In the top row of the grid, the numbers from 1 to 7 are written in order; in every other square, $f(x)$ is written where x is the number above the square. How many functions have the property that the bottom row is identical to the top row, and no other row is identical to the top row?

   Proposed by: Alexander Katz

   Answer: $\boxed{1470}$

   Consider the directed graph with $1, 2, 3, 4, 5, 6, 7$ as vertices, and there is an edge from i to j if and only if $f(i) = j$. Since the bottom row is equivalent to the top one, we have $f^6(x) = x$. Therefore, the graph must decompose into cycles of length 6, 3, 2, or 1. Furthermore, since no other row is equivalent to the top one, the least common multiple of the cycle lengths must be 6. The only partitions of 7 satisfying these constraints are $7 = 6 + 1, 7 = 3 + 2 + 2$, and $7 = 3 + 2 + 1 + 1$.

   If we have a cycle of length 6 and a cycle of length 1, there are 7 ways to choose which six vertices will be in the cycle of length 6, and there are $5! = 120$ ways to determine the values of f within this cycle (to see this, pick an arbitrary vertex in the cycle: the edge from it can connect to any of the remaining 5 vertices, which can connect to any of the remaining 4 vertices, etc.). Hence, there are $7 \cdot 120 = 840$ possible functions f in this case.

   If we have a cycle of length 3 and two cycles of length 2, there are $\frac{\binom{7}{2}\binom{5}{2}}{2} = 105$ possible ways to assign which vertices will belong to which cycle (we divide by two to avoid double-counting the cycles of length 2). As before, there are $2! \cdot 1! \cdot 1! = 2$ assignments of f within the cycles, so there are a total of 210 possible functions f in this case.

Finally, if we have a cycle of length 3, a cycle of length 2, and two cycles of length 1, there are $\binom{7}{3}\binom{4}{2} = 210$ possible ways to assign the cycles, and $2! \cdot 1! \cdot 0! \cdot 0! = 2$ ways to arrange the edges within the cycles, so there are a total of 420 possible functions f in this case.

Hence, there are a total of $840 + 210 + 420 = \boxed{1470}$ possible f.

8. Consider an 8×8 grid of squares. A rook is placed in the lower left corner, and every minute it moves to a square in the same row or column with equal probability (the rook must move; i.e. it cannot stay in the same square). What is the expected number of minutes until the rook reaches the upper right corner?

   Proposed by: Alexander Katz

   Answer: $\boxed{70}$

   Let the expected number of minutes it will take the rook to reach the upper right corner from the top or right edges be E_e, and let the expected number of minutes it will take the rook to reach the upper right corner from any other square be E_c. Note that this is justified because the expected time from any square on the top or right edges is the same, as is the expected time from any other square (this is because swapping any two rows or columns doesn't affect the movement of the rook). This gives us two linear equations:

$$E_c = \frac{2}{14}(E_e + 1) + \frac{12}{14}(E_c + 1)$$
$$E_e = \frac{1}{14}(1) + \frac{6}{14}(E_e + 1) + \frac{7}{14}(E_c + 1)$$

   which gives the solution $E_e = 63, E_c = \boxed{70}$.

9. Consider a 9×9 grid of squares. Haruki fills each square in this grid with an integer between 1 and 9, inclusive. The grid is called a *super-sudoku* if each of the following three conditions hold:

   - Each column in the grid contains each of the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ exactly once.
   - Each row in the grid contains each of the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ exactly once.
   - Each 3×3 subsquare in the grid contains each of the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ exactly once.

   How many possible super-sudoku grids are there?

   Proposed by: Alexander Katz

   Answer: $\boxed{0}$

   Without loss of generality, suppose that the top left corner contains a 1, and examine the top left 3×4:

1	x	x	x
x	x	x	*
x	x	x	*

   There cannot be another 1 in any of the cells marked with an x, but the 3×3 on the right must contain a 1, so one of the cells marked with a * must be a 1. Similarly, looking at the top left 4×3:

1	x	x
x	x	x
x	x	x
x	*	*

   One of the cells marked with a * must also contain a 1. But then the 3×3 square diagonally below the top left one:

1	x	x	x
x	x	x	*
x	x	x	*
x	*	*	?

must contain multiple 1s, which is a contradiction. Hence no such supersudokus exist.

10. Consider a 10×10 grid of squares. One day, Daniel drops a burrito in the top left square, where a wingless pigeon happens to be looking for food. Every minute, if the pigeon and the burrito are in the same square, the pigeon will eat 10% of the burrito's original size and accidentally throw it into a random square (possibly the one it is already in). Otherwise, the pigeon will move to an adjacent square, decreasing the distance between it and the burrito. What is the expected number of minutes before the pigeon has eaten the entire burrito?

    Proposed by: Sam Korsky

    Answer: $\boxed{71.8}$

    Label the squares using coordinates, letting the top left corner be $(0, 0)$. The burrito will end up in 10 (not necessarily different) squares. Call them $p_1 = (x_1, y_1) = (0, 0), p_2 = (x_2, y_2), \ldots, p_{10} = (x_{10}, y_{10})$. p_2 through p_{10} are uniformly distributed throughout the square. Let $d_i = |x_{i+1} - x_i| + |y_{i+1} - y_i|$, the taxicab distance between p_i and p_{i+1}.

    After 1 minute, the pigeon will eat 10% of the burrito. Note that if, after eating the burrito, the pigeon throws it to a square taxicab distance d from the square it's currently in, it will take exactly d minutes for it to reach that square, regardless of the path it takes, and another minute for it to eat 10% of the burrito.

    Hence, the expected number of minutes it takes for the pigeon to eat the whole burrito is

$$\begin{aligned}
1 + E\left(\sum_{i=1}^{9}(d_i + 1)\right) &= 1 + E\left(\sum_{i=1}^{9} 1 + |x_{i+1} - x_i| + |y_{i+1} - y_i|\right) \\
&= 10 + 2 \cdot E\left(\sum_{i=1}^{9} |x_{i+1} - x_i|\right) \\
&= 10 + 2 \cdot \left(E(|x_2|) + E(\sum_{i=2}^{9} |x_{i+1} - x_i|)\right) \\
&= 10 + 2 \cdot (E(|x_2|) + 8 \cdot E(|x_{i+1} - x_i|)) \\
&= 10 + 2 \cdot \left(4.5 + 8 \cdot \frac{1}{100} \cdot \sum_{k=1}^{9} k(20 - 2k)\right) \\
&= 10 + 2 \cdot (4.5 + 8 \cdot 3.3) \\
&= 71.8
\end{aligned}$$