HMMT November 2016

November 12, 2016

General

1. If a and b satisfy the equations $a + \frac{1}{b} = 4$ and $\frac{1}{a} + b = \frac{16}{15}$, determine the product of all possible values of ab.

   Proposed by: Eshaan Nichani

   Answer: $\boxed{1}$

   We multiply $a + \frac{1}{b} = 4$ and $\frac{1}{a} + b = \frac{16}{15}$ to get $2 + ab + \frac{1}{ab} = \frac{64}{15} \iff (ab)^2 - \frac{34}{15}(ab) + 1 = 0$. Since $\left(\frac{34}{14}\right)^2 - 4 \cdot 1 \cdot 1 > 0$, there are two roots ab, so the product of both possible values of ab is $\boxed{1}$ by Vieta's.

2. I have five different pairs of socks. Every day for five days, I pick two socks at random without replacement to wear for the day. Find the probability that I wear matching socks on both the third day and the fifth day.

   Proposed by: Kevin Yang

   Answer: $\boxed{\frac{1}{63}}$

   I get a matching pair on the third day with probability $\frac{1}{9}$ because there is a $\frac{1}{9}$ probability of the second sock matching the first. Given that I already removed a matching pair of the third day, I get a matching pair on the fifth day with probability $\frac{1}{7}$. We multiply these probabilities to get $\boxed{\dfrac{1}{63}}$.

3. Let V be a rectangular prism with integer side lengths. The largest face has area 240 and the smallest face has area 48. A third face has area x, where x is not equal to 48 or 240. What is the sum of all possible values of x?

   Proposed by: Eshaan Nichani

   Answer: $\boxed{260}$

   Let the length, width, and height of the prism be s_1, s_2, s_3. WIthout loss of generality, assume that $s_1 \leq s_2 \leq s_3$. Then, we have that $s_1 s_2 = 48$ and $s_2 s_3 = 240$. Noting that $s_1 \leq s_2$, we must have $(s_1, s_2) = (1, 48), (2, 24), (3, 16), (4, 12), (6, 8)$. We must also have $s_2 s_3 = 240$ and $s_2 \leq s_3$, and the only possibilities for (s_1, s_2) that yield integral s_3 that satisfy these conditions are $(4, 12)$, which gives $s_3 = 20$, and $(6, 8)$, which gives $s_3 = 30$. Thus, the only valid (s_1, s_2, s_3) are $(4, 12, 20)$ and $(6, 8, 30)$. It follows that the only possible areas of the third face are $4(20) = 80$ and $6(30) = 180$, so the desired answer is $80 + 180 = \boxed{260}$.

4. A rectangular pool table has vertices at $(0, 0)$ $(12, 0)$ $(0, 10)$, and $(12, 10)$. There are pockets only in the four corners. A ball is hit from $(0, 0)$ along the line $y = x$ and bounces off several walls before eventually entering a pocket. Find the number of walls that the ball bounces off of before entering a pocket.

   Proposed by: Allen Liu

   Answer: $\boxed{9}$

   Consider the tiling of the plane with the 12×10 rectangle to form a grid. Then the reflection of the ball off a wall is equivalent to traveling along the straight line $y = x$ into another 12×10 rectangle. Hence we want to find the number of walls of the grid that the line $y = x$ hits before it reaches the first corner it hits, $(60, 60)$.

   The line $y = x$ hits each of the horizontal lines $y = 10, 20, 30, 40, 50$ and each of the vertical lines $x = 12, 24, 36, 48$. This gives a total of $\boxed{9}$ walls hit before entering a pocket.

5. Let the sequence $\{a_i\}_{i=0}^{\infty}$ be defined by $a_0 = \frac{1}{2}$ and $a_n = 1 + (a_{n-1} - 1)^2$. Find the product

$$\prod_{i=0}^{\infty} a_i = a_0 a_1 a_2 \ldots$$

Proposed by: Henrik Boecken

Answer: $\boxed{\frac{2}{3}}$

Let $\{b_i\}_{i=0}^{\infty}$ be defined by $b_n = a_n - 1$ and note that $b_n = b_{n-1}^2$. The infinite product is then

$$(1+b_0)(1+b_0^2)(1+b_0^4)\dots(1+b_0^{2^k})\dots$$

By the polynomial identity

$$(1+x)(1+x^2)(1+x^4)\dots(1+x^{2^k})\dots = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

Our desired product is then simply

$$\frac{1}{1-(a_0-1)} = \boxed{\frac{2}{3}}$$

6. The numbers $1, 2 \cdots 11$ are arranged in a line from left to right in a random order. It is observed that the middle number is larger than exactly one number to its left. Find the probability that it is larger than exactly one number to its right.

   Proposed by: Allen Liu

   Answer: $\boxed{\frac{10}{33}}$

   Suppose the middle number is k. Then there are $k-1$ ways to pick the number smaller than k to its left and $\binom{11-k}{4}$ ways to pick the 4 numbers larger than k to its right. Hence there is a total of $\sum_{k=2}^{7}(k-1)\cdot\binom{11-k}{4}$ ways for there to be exactly one number smaller than k to its left. We calculate this total:

$$\begin{aligned}\sum_{k=2}^{7}(k-1)\cdot\binom{11-k}{4} &= \sum_{j=4}^{9}\sum_{i=4}^{j}\binom{i}{4} \\ &= \sum_{j=4}^{9}\binom{j+1}{5} \\ &= \binom{11}{6}.\end{aligned}$$

   The only way k can be larger than exactly one number to its right is if $k = 3$. Then the probability of this happening is $\frac{2\cdot\binom{8}{4}}{\binom{11}{6}} = \boxed{\frac{10}{33}}$.

7. Let ABC be a triangle with $AB = 13$, $BC = 14$, $CA = 15$. The altitude from A intersects BC at D. Let ω_1 and ω_2 be the incircles of ABD and ACD, and let the common external tangent of ω_1 and ω_2 (other than BC) intersect AD at E. Compute the length of AE.

   Proposed by: Eshaan Nichani

   Answer: $\boxed{7}$

   Solution 1:

   Let I_1, I_2 be the centers of ω_1, ω_2, respectively, X_1, X_2 be the tangency points of ω_1, ω_2 with BC, respectively, and Y_1, Y_2 be the tangency points of ω_1, ω_2 with AD, respectively. Let the two common external tangents of ω_1, ω_2 meet at P. Note that line I_1I_2 also passes through P.

   By Heron's formula, the area of triangle ABC is 84. Thus, $\frac{1}{2}AD \cdot BC = 84$, and so $AD = 12$. By the Pythagorean Theorem on right triangles ABD and ACD, $BD = 5$ and $CD = 9$.

   The inradius of ABD, r_{ABD}, is $\frac{[ABD]}{s_{ABD}}$, where $[ABD]$ is the area of ABD and s_{ABD} is its semiperimeter. ABD is a 5-12-13 right triangle, so $[ABD] = 30$ and $s_{ABD} = 15$. Thus, $r_{ABD} = 2$. Similarly, we get that ACD's inradius is $r_{ACD} = 3$. $I_1Y_1DX_1$ is a square, so $X_1D = I_1X_1 = r_{ABD} = 2$, and similarly $X_2D = 3$. X_1 and X_2 are on opposite sides of D, so $X_1X_2 = X_1D + X_2D = 5$.

Since P lies on lines I_1I_2, X_1X_2, and I_1X_1, I_2X_2 are parallel, triangles PX_1I_1 and PX_2I_2 are similar. Thus, $\frac{X_1I_1}{X_2I_2} = \frac{2}{3} = \frac{PX_1}{PX_2} = \frac{PX_1}{PX_1+X_1X_2} = \frac{PX_1}{PX_1+5}$. Solving gives $PX_1 = 10$. Letting $\angle I_1PX_1 = \theta$, since I_1X_1P is a right angle, we have $\tan\theta = \frac{X_1I_1}{X_1P_1} = \frac{1}{5}$. D and E lie on different common external tangents, so PI_1 bisects $\angle EPD$, and thus $\angle EPD = 2\theta$. Thus, $\tan\angle EPD = \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{5}{12}$.

ED is perpendicular to BC, so triangle EDP is a right triangle with right angle at D. Thus, $\frac{5}{12} = \tan\angle EPD = \frac{ED}{PD}$. $PD = PX_1 + X_1D = 12$, so $ED = 5$. $AD = 12$, so it follows that $AE = \boxed{7}$.

Solution 2:

Lemma: Let Ω_1, Ω_2 be two non-intersecting circles. Let ℓ_A, ℓ_B be their common external tangents, and ℓ_C be one of their common internal tangents. Ω_1 intersects ℓ_A, ℓ_B at points A_1, B_1, respectively, and Ω_2 intersects ℓ_A, ℓ_B at points A_2, B_2. If ℓ_C intersects ℓ_A at X, and ℓ_B at Y, then $XY = A_1A_2 = B_1B_2$.

Proof: Let ℓ_C intersect Ω_1, Ω_2 at C_1, C_2, respectively. Then, examining the tangents to Ω_1, Ω_2 from X, Y, we have $A_1X = C_1X, A_2X = C_2X, B_1Y = C_1Y, B_2Y = C_2Y$. Thus, $2A_1A_2 = 2B_1B_2 = A_1A_2 + B_1B_2 = A_1X + A_2X + B_1Y + B_2Y = C_1X + C_2X + C_1Y + C_2Y = 2XY$, and the conclusion follows.

Using the notation from above, we apply the lemma to circles ω_1, ω_2, and conclude that $ED = X_1X_2$. Then, we proceed as above to compute $X_1X_2 = 5 = ED$. Thus, $AD = \boxed{7}$.

8. Let $S = \{1, 2, \ldots 2016\}$, and let f be a randomly chosen bijection from S to itself. Let n be the smallest positive integer such that $f^{(n)}(1) = 1$, where $f^{(i)}(x) = f(f^{(i-1)}(x))$. What is the expected value of n?

   Proposed by: Eshaan Nichani

   Answer: $\boxed{\frac{2017}{2}}$

   Say that $n = k$. Then $1, f(1), f^2(1), \ldots, f^{(k-1)}(1)$ are all distinct, which means there are $2015 \cdot 2014 \cdots (2016 - k + 1)$ ways to assign these values. There is 1 possible value of $f^k(1)$, and $(2016 - k)!$ ways to assign the image of the $2016 - k$ remaining values. Thus the probability that $n = k$ is $\frac{1}{2016}$. Therefore the expected value of n is $\frac{1}{2016}(1 + 2 + \cdots + 2016) = \frac{2017}{2}$

9. Let the sequence a_i be defined as $a_{i+1} = 2^{a_i}$. Find the number of integers $1 \le n \le 1000$ such that if $a_0 = n$, then 100 divides $a_{1000} - a_1$.

   Proposed by: Allen Liu

   Answer: $\boxed{50}$

   We claim that a_{1000} is constant mod 100.

   a_{997} is divisible by 2. This means that a_{998} is divisible by 4. Thus a_{999} is constant mod 5. Since it is also divisible by 4, it is constant mod 20. Thus a_{1000} is constant mod 25, since $\phi(25) = 20$. Since a_{1000} is also divisible by 4, it is constant mod 100.

   We know that a_{1000} is divisible by 4, and let it be congruent to k mod 25.

   Then 2^n is divisible by 4 $(n \ge 2)$ and $2^n \equiv k$ mod 25 We can also show that 2 is a primitive root mod 25, so there is one unique value of n mod 20. It suffices to show this value isn't 1. But $2^{2^{0 mod 4}} \equiv 2^{16 mod 20}$ mod 25, so $n \equiv 16$ mod 20. Thus there are $1000/20 = 50$ values of n.

10. Quadrilateral $ABCD$ satisfies $AB = 8$, $BC = 5$, $CD = 17$, $DA = 10$. Let E be the intersection of AC and BD. Suppose $BE : ED = 1 : 2$. Find the area of $ABCD$.

    Proposed by: Brice Huang

    Answer: $\boxed{60}$

    Since $BE : ED = 1 : 2$, we have $[ABC] : [ACD] = 1 : 2$.

    Suppose we cut off triangle ACD, reflect it across the perpendicular bisector of AC, and re-attach it as triangle $A'C'D'$ (so $A' = C, C' = A$).

Triangles ABC and $C'A'D'$ have vertex $A = C'$ and bases BC and $A'D'$. Their areas and bases are both in the ratio $1 : 2$. Thus in fact BC and $A'D'$ are collinear.

Hence the union of ABC and $C'A'D'$ is the $8 - 15 - 17$ triangle ABD', which has area $\frac{1}{2} \cdot 8 \cdot 15 = 60$.