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1. [**5**] If five fair coins are flipped simultaneously, what is the probability that at least three of them show heads?

2. [**5**] How many perfect squares divide 10^{10}?

3. [**5**] Evaluate $\frac{2016!^2}{2015!2017!}$. Here $n!$ denotes $1 \times 2 \times \cdots \times n$.

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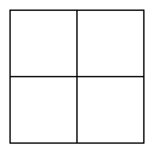
4. [**6**] A square can be divided into four congruent figures as shown:



For how many n with $1 \leq n \leq 100$ can a unit square be divided into n congruent figures?

5. [**6**] If $x + 2y - 3z = 7$ and $2x - y + 2z = 6$, determine $8x + y$.

6. [**6**] Let $ABCD$ be a rectangle, and let E and F be points on segment AB such that $AE = EF = FB$. If CE intersects the line AD at P, and PF intersects BC at Q, determine the ratio of BQ to CQ.

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7. [**7**] What is the minimum value of the product

$$\prod_{i=1}^{6} \frac{a_i - a_{i+1}}{a_{i+2} - a_{i+3}}$$

given that $(a_1, a_2, a_3, a_4, a_5, a_6)$ is a permutation of $(1, 2, 3, 4, 5, 6)$? (note $a_7 = a_1, a_8 = a_2 \cdots$)

8. [**7**] Danielle picks a positive integer $1 \leq n \leq 2016$ uniformly at random. What is the probability that $\gcd(n, 2015) = 1$?

9. [**7**] How many 3-element subsets of the set $\{1, 2, 3, \ldots, 19\}$ have sum of elements divisible by 4?

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10. [**8**] Michael is playing basketball. He makes 10% of his shots, and gets the ball back after 90% of his missed shots. If he does not get the ball back he stops playing. What is the probability that Michael eventually makes a shot?

11. [**8**] How many subsets S of the set $\{1, 2, \ldots, 10\}$ satisfy the property that, for all $i \in [1, 9]$, either i or $i + 1$ (or both) is in S?

12. [**8**] A positive integer $\overline{ABC}$, where A, B, C are digits, satisfies

$$\overline{ABC} = B^C - A$$

Find $\overline{ABC}$.

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13. [**9**] How many functions $f : \{0,1\}^3 \to \{0,1\}$ satisfy the property that, for all ordered triples (a_1, a_2, a_3) and (b_1, b_2, b_3) such that $a_i \geq b_i$ for all i, $f(a_1, a_2, a_3) \geq f(b_1, b_2, b_3)$?

14. [**9**] The very hungry caterpillar lives on the number line. For each non-zero integer i, a fruit sits on the point with coordinate i. The caterpillar moves back and forth; whenever he reaches a point with food, he eats the food, increasing his weight by one pound, and turns around. The caterpillar moves at a speed of 2^{-w} units per day, where w is his weight. If the caterpillar starts off at the origin, weighing zero pounds, and initially moves in the positive x direction, after how many days will he weigh 10 pounds?

15. [**9**] Let $ABCD$ be an isosceles trapezoid with parallel bases $AB = 1$ and $CD = 2$ and height 1. Find the area of the region containing all points inside $ABCD$ whose projections onto the four sides of the trapezoid lie on the segments formed by AB,BC,CD and DA.

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16. [**10**] Create a cube C_1 with edge length 1. Take the centers of the faces and connect them to form an octahedron O_1. Take the centers of the octahedron's faces and connect them to form a new cube C_2. Continue this process infinitely. Find the sum of all the surface areas of the cubes and octahedrons.

17. [**10**] Let $p(x) = x^2 - x + 1$. Let α be a root of $p(p(p(p(x))))$. Find the value of

$$(p(\alpha) - 1)p(\alpha)p(p(\alpha))p(p(p(\alpha))$$

18. [**10**] An 8 by 8 grid of numbers obeys the following pattern:

1) The first row and first column consist of all 1s.

2) The entry in the ith row and jth column equals the sum of the numbers in the $(i-1)$ by $(j-1)$ sub-grid with row less than i and column less than j.

What is the number in the 8th row and 8th column?

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19. [**11**] Let S be the set of all positive integers whose prime factorizations only contain powers of the primes 2 and 2017 (1, powers of 2, and powers of 2017 are thus contained in S). Compute $\sum_{s\in S} \frac{1}{s}$.

20. [**11**] Let $\mathcal{V}$ be the volume enclosed by the graph

$$x^{2016} + y^{2016} + z^2 = 2016$$

Find $\mathcal{V}$ rounded to the nearest multiple of ten.

21. [**11**] Zlatan has 2017 socks of various colours. He wants to proudly display one sock of each of the colours, and he counts that there are N ways to select socks from his collection for display. Given this information, what is the maximum value of N?

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22. [**12**] Let the function $f : \mathbb{Z} \to \mathbb{Z}$ take only integer inputs and have integer outputs. For any integers x and y, f satisfies

$$f(x) + f(y) = f(x + 1) + f(y - 1)$$

If $f(2016) = 6102$ and $f(6102) = 2016$, what is $f(1)$?

23. [**12**] Let d be a randomly chosen divisor of 2016. Find the expected value of

$$\frac{d^2}{d^2 + 2016}$$

24. [**12**] Consider an infinite grid of equilateral triangles. Each edge (that is, each side of a small triangle) is colored one of N colors. The coloring is done in such a way that any path between any two non-adjecent vertices consists of edges with at least two different colors. What is the smallest possible value of N?

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25. [**13**] Chris and Paul each rent a different room of a hotel from rooms $1-60$. However, the hotel manager mistakes them for one person and gives "Chris Paul" a room with Chris's and Paul's room concatenated. For example, if Chris had 15 and Paul had 9, "Chris Paul" has 159. If there are 360 rooms in the hotel, what is the probability that "Chris Paul" has a valid room?

26. [**13**] Find the number of ways to choose two nonempty subsets X and Y of $\{1, 2, \ldots, 2001\}$, such that $|Y| = 1001$ and the smallest element of Y is equal to the largest element of X.

27. [**13**] Let r_1, r_2, r_3, r_4 be the four roots of the polynomial $x^4 - 4x^3 + 8x^2 - 7x + 3$. Find the value of

$$\frac{r_1^2}{r_2^2+r_3^2+r_4^2}+\frac{r_2^2}{r_1^2+r_3^2+r_4^2}+\frac{r_3^2}{r_1^2+r_2^2+r_4^2}+\frac{r_4^2}{r_1^2+r_2^2+r_3^2}$$

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28. [**15**] The numbers $1-10$ are written in a circle randomly. Find the expected number of numbers which are at least 2 larger than an adjacent number.

29. [**15**] We want to design a new chess piece, the American, with the property that (i) the American can never attack itself, and (ii) if an American A_1 attacks another American A_2, then A_2 also attacks A_1. Let m be the number of squares that an American attacks when placed in the top left corner of an 8 by 8 chessboard. Let n be the maximal number of Americans that can be placed on the 8 by 8 chessboard such that no Americans attack each other, if one American must be in the top left corner. Find the largest possible value of mn.

30. [**15**] On the blackboard, Amy writes 2017 in base-a to get 133201_a. Betsy notices she can erase a digit from Amy's number and change the base to base-b such that the value of the the number remains the same. Catherine then notices she can erase a digit from Betsy's number and change the base to base-c such that the value still remains the same. Compute, in decimal, $a + b + c$.

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31. [**17**] Define a number to be an anti-palindrome if, when written in base 3 as $a_n a_{n-1} \ldots a_0$, then $a_i + a_{n-i} = 2$ for any $0 \leq i \leq n$. Find the number of anti-palindromes less than 3^{12} such that no two consecutive digits in base 3 are equal.

32. [**17**] Let $C_{k,n}$ denote the number of paths on the Cartesian plane along which you can travel from $(0,0)$ to (k,n), given the following rules: 1) You can only travel directly upward or directly rightward 2) You can only change direction at lattice points 3) Each horizontal segment in the path must be at most 99 units long.

    Find
    $$\sum_{j=0}^{\infty} C_{100j+19,17}$$

33. [**17**] Camille the snail lives on the surface of a regular dodecahedron. Right now he is on vertex P_1 of the face with vertices P_1, P_2, P_3, P_4, P_5. This face has a perimeter of 5. Camille wants to get to the point on the dodecahedron farthest away from P_1. To do so, he must travel along the surface a distance at least L. What is L^2?

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34. [**20**] Find the sum of the ages of everyone who wrote a problem for this year's HMMT November contest. If your answer is X and the actual value is Y, your score will be $\max(0, 20 - |X - Y|)$

35. [**20**] Find the total number of occurrences of the digits $0, 1 \ldots, 9$ in the entire guts round. If your answer is X and the actual value is Y, your score will be $\max(0, 20 - \frac{|X-Y|}{2})$

36. [**20**] Find the number of positive integers less than 1000000 which are less than or equal to the sum of their proper divisors. If your answer is X and the actual value is Y, your score will be $\max(0, 20 - 80|1 - \frac{X}{Y}|)$ rounded to the nearest integer.