HMMT November 2016

November 12, 2016

Team

1. [**3**] Two circles centered at O_1 and O_2 have radii 2 and 3 and are externally tangent at P. The common external tangent of the two circles intersects the line O_1O_2 at Q. What is the length of PQ?

2. [**3**] What is the smallest possible perimeter of a triangle whose side lengths are all squares of distinct positive integers?

3. [**3**] Complex number ω satisfies $\omega^5 = 2$. Find the sum of all possible values of
$$\omega^4 + \omega^3 + \omega^2 + \omega + 1.$$

4. [**5**] Meghal is playing a game with 2016 rounds $1, 2, \cdots, 2016$. In round n, two rectangular double-sided mirrors are arranged such that they share a common edge and the angle between the faces is $\frac{2\pi}{n+2}$. Meghal shoots a laser at these mirrors and her score for the round is the number of points on the two mirrors at which the laser beam touches a mirror. What is the maximum possible score Meghal could have after she finishes the game?

5. [**5**] Allen and Brian are playing a game in which they roll a 6-sided die until one of them wins. Allen wins if two consecutive rolls are equal and at most 3. Brian wins if two consecutive rolls add up to 7 and the latter is at most 3. What is the probability that Allen wins?

6. [**5**] Let ABC be a triangle with $AB = 5$, $BC = 6$, and $AC = 7$. Let its orthocenter be H and the feet of the altitudes from A, B, C to the opposite sides be D, E, F respectively. Let the line DF intersect the circumcircle of AHF again at X. Find the length of EX.

7. [**6**] Rachel has two indistinguishable tokens, and places them on the first and second square of a 1×6 grid of squares, She can move the pieces in two ways:

   - If a token has free square in front of it, then she can move this token one square to the right
   - If the square immediately to the right of a token is occupied by the other token, then she can"leapfrog" the first token; she moves the first token two squares to the right, over the other token, so that it is on the square immediately to the right of the other token.

   If a token reaches the 6th square, then it cannot move forward any more, and Rachel must move the other one until it reaches the 5th square. How many different sequences of moves for the tokens can Rachel make so that the two tokens end up on the 5th square and the 6th square?

8. [**6**] Alex has an 20×16 grid of lightbulbs, initially all off. He has 36 switches, one for each row and column. Flipping the switch for the ith row will toggle the state of each lightbulb in the ith row (so that if it were on before, it would be off, and vice versa). Similarly, the switch for the jth column will toggle the state of each bulb in the jth column. Alex makes some (possibly empty) sequence of switch flips, resulting in some configuration of the lightbulbs and their states. How many distinct possible configurations of lightbulbs can Alex achieve with such a sequence? Two configurations are distinct if there exists a lightbulb that is on in one configuration and off in another.

9. [**7**] A cylinder with radius 15 and height 16 is inscribed in a sphere. Three congruent smaller spheres of radius x are externally tangent to the base of the cylinder, externally tangent to each other, and internally tangent to the large sphere. What is the value of x?

10. [**7**] Determine the largest integer n such that there exist monic quadratic polynomials $p_1(x)$, $p_2(x)$, $p_3(x)$ with integer coefficients so that for all integers $i \in [1, n]$ there exists some $j \in [1, 3]$ and $m \in \mathbb{Z}$ such that $p_j(m) = i$.