HMMT November 2016

November 12, 2016

Team

1. [**3**] Two circles centered at O_1 and O_2 have radii 2 and 3 and are externally tangent at P. The common external tangent of the two circles intersects the line O_1O_2 at Q. What is the length of PQ?

   Proposed by: Eshaan Nichani

   Answer: $\boxed{12}$

   Let the common external tangent intersect the circles centered at O_1, O_2 at X, Y respectively. Then $\frac{O_2Q}{O_1Q} = \frac{OY}{OX} = \frac{3}{2}$, so $\frac{O_1O_2}{O_1Q} = \frac{O_2Q - O_1Q}{O_1Q} = \frac{1}{2}$. Since $O_1O_2 = 2 + 3 = 5$, $O_1Q = 10$ and hence $PQ = O_1Q + O_1P = 12$.

2. [**3**] What is the smallest possible perimeter of a triangle whose side lengths are all squares of distinct positive integers?

   Proposed by: Eshaan Nichani

   Answer: $\boxed{77}$

   There exist a triangle with side lengths $4^2, 5^2, 6^2$, which has perimeter 77. If the sides have lengths a^2, b^2, c^2 with $0 < a < b < c$, then $a^2 + b^2 > c^2$ by the triangle inequality. Therefore $(b-1)^2 + b^2 \geq a^2 + b^2 > c^2 \geq (b+1)^2$. Solving this inequality gives $b > 4$. If $b \geq 6$, then $a^2 + b^2 + c^2 \geq 6^2 + 7^2 > 77$. If $b = 5$, then $c \geq 7$ is impossible, while $c = 6$ forces $a = 4$, which gives a perimeter of $\boxed{77}$.

3. [**3**] Complex number ω satisfies $\omega^5 = 2$. Find the sum of all possible values of

   $$\omega^4 + \omega^3 + \omega^2 + \omega + 1.$$

   Proposed by: Henrik Boecken

   Answer: $\boxed{5}$

   The value of $\omega^4 + \omega^3 + \omega^2 + \omega + 1 = \dfrac{\omega^5 - 1}{\omega - 1} = \dfrac{1}{\omega - 1}$. The sum of these values is therefore the sum of $\dfrac{1}{\omega - 1}$ over the five roots ω. Substituting $z = \omega - 1$, we have that $(z+1)^5 = 2$, so $z^5 + 5z^4 + 10z^3 + 10z^2 + 5z - 1 = 0$. The sum of the reciprocals of the roots of this equation is $-\dfrac{5}{-1} = \boxed{5}$ by Vieta's.

4. [**5**] Meghal is playing a game with 2016 rounds $1, 2, \cdots, 2016$. In round n, two rectangular double-sided mirrors are arranged such that they share a common edge and the angle between the faces is $\frac{2\pi}{n+2}$. Meghal shoots a laser at these mirrors and her score for the round is the number of points on the two mirrors at which the laser beam touches a mirror. What is the maximum possible score Meghal could have after she finishes the game?

   Proposed by: Rachel Zhang

   Answer: $\boxed{1019088}$

   Let points O, A_1, A_2 lie in a plane such that $\angle A_1OA_2 = \frac{2\pi}{n+2}$. We represent the mirrors as line segments extending between O and A_1, and O and A_2. Also let points $A_3, A_4, \cdots, A_{n+2}$ lie in the plane such that A_{i+1} is the reflection of A_{i-1} over OA_i.

   If Meghal shoots a laser along line l such that the first point of contact with a mirror is along OA_2, the next point of contact, if it exists, is the point on OA_1 that is a reflection of the intersection of l with OA_3. If we continue this logic, we find that the maximum score for round n is equal to the maximum number of intersection points between l and OA_i for some i. We do casework on whether n is even or odd. If n is even, there are at most $\frac{n+2}{2}$ spokes such that l can hit OA_i, and if n is odd, there are at most $\frac{n+3}{2}$ such spokes. Then we must sum $2 + 2 + 3 + 3 + \cdots + 1009 + 1009 = 1009 \cdot 1010 - 1 - 1 = \boxed{1019088}$.

5. [**5**] Allen and Brian are playing a game in which they roll a 6-sided die until one of them wins. Allen wins if two consecutive rolls are equal and at most 3. Brian wins if two consecutive rolls add up to 7 and the latter is at most 3. What is the probability that Allen wins?

Proposed by: Eshaan Nichani

Answer: $\boxed{5/12}$

Note that at any point in the game after the first roll, the probability that Allen wins depends only on the most recent roll, and not on any rolls before that one. So we may define p as the probability that Allen wins at any point in the game, given that the last roll was a 1, 2, or 3, and q as the probability that he wins given that the last roll was a 4, 5, or 6.

Suppose at some point, the last roll was $r_1 \in \{1, 2, 3\}$, and the next roll is $r_2 \in \{1, 2, 3, 4, 5, 6\}$. By the definition of p, Allen wins with probability p. Furthermore, if $r_2 = r_1$, which happens with probability $\frac{1}{6}$, Allen wins. If $r_2 \in \{1, 2, 3\}$ but $r_2 \neq r_1$, which happens with probability $\frac{2}{6}$, neither Allen nor Brian wins, so they continue playing the game, now where the last roll was r_2. In this case, Allen wins with probability p. If $r_2 \in \{4, 5, 6\}$, which happens with probability $\frac{3}{6}$, neither Allen nor Brian wins, so they continue playing, now where the last roll was r_2. In this case, Allen wins with probability q. Hence, the probability that Allen wins in this case can be expressed as $\frac{1}{6} + \frac{2}{6}p + \frac{3}{6}q$, and thus

$$p = \frac{1}{6} + \frac{2}{6}p + \frac{3}{6}q$$

By a similar analysis for q, we find that

$$q = \frac{1}{6} \cdot 0 + \frac{2}{6}p + \frac{3}{6}q$$

Solving, we get $p = \frac{1}{2}$ and $q = \frac{1}{3}$. Allen wins with probability $p = \frac{1}{2}$ if the first roll is 1, 2, or 3, and he wins with probability $q = \frac{1}{3}$ if the first roll is 4, 5, or 6. We conclude that the overall probability that he wins the game is $\frac{1}{2}p + \frac{1}{2}q = \boxed{\frac{5}{12}}$.

6. [**5**] Let ABC be a triangle with $AB = 5$, $BC = 6$, and $AC = 7$. Let its orthocenter be H and the feet of the altitudes from A, B, C to the opposite sides be D, E, F respectively. Let the line DF intersect the circumcircle of AHF again at X. Find the length of EX.

Proposed by: Allen Liu

Answer: $\boxed{\frac{190}{49}}$

Since $\angle AFH = \angle AEH = 90°$, E is on the circumcircle of AHF. So $\angle XEH = \angle HFD = \angle HBD$, which implies that $XE \parallel BD$. Hence $\frac{EX}{BD} = \frac{EY}{YB}$. Let DF and BE intersect at Y. Note that $\angle EDY = 180° - \angle BDF - \angle CDE = 180° - 2\angle A$, and $\angle BDY = \angle A$. Applying the sine rule to EYD and BYD, we get

$$\frac{EY}{YB} = \frac{ED}{BD} \cdot \frac{\sin \angle EDY}{\sin \angle BDY} = \frac{ED}{BD} \cdot \frac{\sin 2\angle A}{\sin \angle A} = \frac{ED}{BD} \cdot 2\cos \angle A$$

Next, letting $x = CD$ and $y = AE$, by Pythagoras we have

$$AB^2 - (6 - x)^2 = AD^2 = AC^2 - x^2$$

$$BC^2 - (7 - y)^2 = BE^2 = BA^2 - y^2$$

Solving, we get $x = 5, y = \frac{19}{7}$. Drop the perpendicular from E to DC at Z. Then $ED \cos \angle A = ED \cos \angle EDZ = DZ$. But $AD \parallel EZ$, so $DZ = \frac{AE}{AC} \cdot DC = \frac{95}{49}$. Therefore

$$EX = \frac{EY}{YB} \cdot BD = 2ED \cos \angle A = 2DZ = \frac{190}{49}$$

7. [**6**] Rachel has two indistinguishable tokens, and places them on the first and second square of a 1×6 grid of squares, She can move the pieces in two ways:

   - If a token has free square in front of it, then she can move this token one square to the right
   - If the square immediately to the right of a token is occupied by the other token, then she can"leapfrog" the first token; she moves the first token two squares to the right, over the other token, so that it is on the square immediately to the right of the other token.

   If a token reaches the 6th square, then it cannot move forward any more, and Rachel must move the other one until it reaches the 5th square. How many different sequences of moves for the tokens can Rachel make so that the two tokens end up on the 5th square and the 6th square?

   Proposed by: Christopher Shao

   Answer: 42

   We put a marker on (i, j) when a token is on ith and jth square and $i > j$. When the token in front/behind moves one step forward to a blank square, move the marker rightward/upward one unit correspondingly. When a "leapfrog" happens, the marker moves from $(x-1, x)$ to $(x, x+1)$. We can translate this movement into: 1. move the marker upward to (x, x); 2. move the marker rightward to $(x, x+1)$. Thus, we set up a lattice path way from $(2, 1)$ to $(6, 5)$ staying under $y = x$. This is a bijection since every intersection of the path way and $y = x$ indicates a "leapfrog". According to the definition of Catalan Number, the answer is the number of such lattice path ways, which is $C_5 = 42$.

8. [**6**] Alex has an 20×16 grid of lightbulbs, initially all off. He has 36 switches, one for each row and column. Flipping the switch for the ith row will toggle the state of each lightbulb in the ith row (so that if it were on before, it would be off, and vice versa). Similarly, the switch for the jth column will toggle the state of each bulb in the jth column. Alex makes some (possibly empty) sequence of switch flips, resulting in some configuration of the lightbulbs and their states. How many distinct possible configurations of lightbulbs can Alex achieve with such a sequence? Two configurations are distinct if there exists a lightbulb that is on in one configuration and off in another.

   Proposed by: Christopher Shao

   Answer: 2^{35}

   The switch flip operations are commutative, so for any given sequence of switch flips S, we get the same configuration regardless of the order we do them in. We can arrange the switch flips so that all of the flips of the same switch happen consecutively. Furthermore, two consecutive flips of the same switch leave the configuration unchanged, so we can remove them, resulting in a sequence of switch flips S' that involves flipping a switch for a row or column at most once that achieves the same configuration as S. The order of the flips in S' also doesn't matter, so we can treat S' as a set of switches that are flipped to produce the same configuration as S.

   The desired number is then equal to the number of distinct configurations that can be obtained by flipping exactly the switches in some subset S' of the set of all of the switches. We claim that if S_1 and S_2 are distinct sets of switches that result in the same configuration of lights, then S_1 and S_2 are complements. Indeed, without loss of generality, suppose that the first row's switch is in S_1 and that it isn't in S_2. In order to have the same configuration of lights in the first row, we must have that every column switch is in S_1 if and only if it isn't in S_2. Applying the same argument to the first column yields that every row switch is in S_1 if and only if it isn't in S_2, and the claim follows. Thus, for every set of switches, there is exactly one other set that attains the same configuration as it, namely its complement. There are 2^{m+n} sets of switches possible, and so the total number of possible configurations is $2^{m+n}/2 = 2^{m+n-1}$.

9. [**7**] A cylinder with radius 15 and height 16 is inscribed in a sphere. Three congruent smaller spheres of radius x are externally tangent to the base of the cylinder, externally tangent to each other, and internally tangent to the large sphere. What is the value of x?

   Proposed by: Eshaan Nichani

Answer: $\boxed{\frac{15\sqrt{37}-75}{4}}$

Let O be the center of the large sphere, and let O_1, O_2, O_3 be the centers of the small spheres. Consider G, the center of equilateral $\triangle O_1O_2O_3$. Then if the radii of the small spheres are r, we have that $OG = 8+r$ and $O_1O_2 = O_2O_3 = O_3O_1 = 2r$, implying that $O_1G = \frac{2r}{\sqrt{3}}$. Then $OO_1 = \sqrt{OG^2 + OO_1^2} = \sqrt{(8+r)^2 + \frac{4}{3}r^2}$. Now draw the array OO_1, and suppose it intersects the large sphere again at P. Then P is the point of tangency between the large sphere and the small sphere with center O_1, so $OP = \sqrt{15^2 + 8^2} = 17 = OO_1 + O_1P = \sqrt{(8+r)^2 + \frac{4}{3}r^2} + r$. We rearrange this to be

$$\begin{aligned}
17 - r &= \sqrt{(8+r)^2 + \frac{4}{3}r^2} \\
\iff 289 - 34r + r^2 &= \frac{7}{3}r^2 + 16r + 64 \\
\iff \frac{4}{3}r^2 + 50r - 225 &= 0 \\
\implies r &= \frac{-50 \pm \sqrt{50^2 + 4 \cdot \frac{4}{3} \cdot 225}}{2 \cdot \frac{4}{3}} \\
&= \boxed{\frac{15\sqrt{37} - 75}{4}}.
\end{aligned}$$

10. [**7**] Determine the largest integer n such that there exist monic quadratic polynomials $p_1(x)$, $p_2(x)$, $p_3(x)$ with integer coefficients so that for all integers $i \in [1, n]$ there exists some $j \in [1, 3]$ and $m \in \mathbb{Z}$ such that $p_j(m) = i$.

Proposed by: Eshaan Nichani

Answer: $\boxed{9}$

The construction for $n = 9$ can be achieved with the polynomials $x^2 + x + 1$, $x^2 + x + 2$, and $x^2 + 5$.

First we consider what kinds of polynomials we can have. Let $p(x) = (x + h)^2 + k$. h is either an integer or half an integer. Let $k = 0$. If h is an integer then $p(x)$ hits the perfect squares 0, 1, 4, 9, etc. If h is half an integer, then let $k = 1/4$. Then $p(x)$ hits the product of two consecutive integers, i.e. 0, 2, 6, 12, etc.

Assume there is a construction for $n = 10$. In both of the cases above, the most a polynomial can hit out of 10 is 4, in the $0, 1, 4, 9$ case. Thus p_1 must hit $1, 2, 5, 10$, and p_2 and p_3 hit 3 integers each, out of $3, 4, 6, 7, 8, 9$. The only ways we can hit 3 out of 7 consecutive integers is with the sequences $0, 2, 6$ or $0, 1, 4$. The only way a $0, 2, 6$ works is if it hits 3, 5, and 9, which doesn't work since 5 was hit by p_2. Otherwise, p_2 is $0, 1, 4$, which doesn't work as p_2 hits 3, 4, and 7, and p_3 must hit 6, 8, and 9, which is impossible. Thus no construction for $n = 10$ exists.