HMMT November 2017

November 11, 2017

General Round

1. Find the sum of all positive integers whose largest proper divisor is 55. (A proper divisor of n is a divisor that is strictly less than n.)

   Proposed by: Michael Tang

   Answer: $\boxed{550}$

   The largest proper divisor of an integer n is $\frac{n}{p}$, where p is the smallest prime divisor of n. So $n = 55p$ for some prime p. Since $55 = 5 \cdot 11$, we must have $p \leq 5$, so $p = 2, 3, 5$ gives all solutions. The sum of these solutions is $55(2 + 3 + 5) = 550$.

2. Determine the sum of all distinct real values of x such that

   $$|||\cdots||x| + x|\cdots| + x| + x| = 1,$$

   where there are 2017 x's in the equation.

   Proposed by: Yuan Yao

   Answer: $\boxed{-\frac{2016}{2017}}$

   Note that $|x + |x|| = 2x$ when x is nonnegative, and is equal to 0 otherwise. Thus, when there are 2017 x's, the expression equals $2017x$ when $x \geq 0$ and $-x$ otherwise, so the two solutions to the equation are $x = -1$ and $\frac{1}{2017}$, and their sum is $-\frac{2016}{2017}$.

3. Find the number of integers n with $1 \leq n \leq 2017$ so that $(n - 2)(n - 0)(n - 1)(n - 7)$ is an integer multiple of 1001.

   Proposed by: Dhruv Rohatgi

   Answer: $\boxed{99}$

   Note that $1001 = 7 \cdot 11 \cdot 13$, so the stated product must be a multiple of 7, as well as a multiple of 11, as well as a multiple of 13. There are 4 possible residues of n modulo 11 for which the product is a multiple of 11; similarly, there are 4 possible residues of n modulo 13 for which the product is a multiple of 13. However, there are only 3 possible residues of n modulo 7 for which the product is a multiple of 7.

   Consider each of these $4 \cdot 4 \cdot 3 = 48$ possible triples of remainders. By the Chinese Remainder Theorem there is exactly one value of n with $1 \leq n \leq 1001$ achieving those remainders, and exactly one value of n with $16 \leq n \leq 1016$ achieving those remainders. Similarly, there is exactly one value of n with $1017 \leq n \leq 2017$ with those same remainders. Hence there are 96 values of n with $16 \leq n \leq 2017$ such that $(n - 2)(n - 0)(n - 1)(n - 7)$ is a multiple of 1001.

   It remains to check $n \in \{1, 2, 3, \ldots, 15\}$. Since the product must be a multiple of 7, we can narrow the set to $\{1, 2, 7, 8, 9, 14\}$. The first 3 values work trivially, since the product is 0. It can be easily checked that none of the remaining values of n yield a product which is a multiple of 11. Hence, the final answer is $96 + 3 = 99$.

4. Triangle ABC has $AB = 10$, $BC = 17$, and $CA = 21$. Point P lies on the circle with diameter AB. What is the greatest possible area of APC?

   Proposed by: Michael Tang

   Answer: $\boxed{\frac{189}{2}}$

   To maximize $[APC]$, point P should be the farthest point on the circle from AC. Let M be the midpoint of AB and Q be the projection of M onto AC. Then $PQ = PM + MQ = \frac{1}{2}AB + \frac{1}{2}h_B$, where h_B is the length of the altitude from B to AC. By Heron's formula, one finds that the area of ABC is $\sqrt{24 \cdot 14 \cdot 7 \cdot 3} = 84$, so $h_B = \frac{2 \cdot 84}{AC} = 8$. Then $PQ = \frac{1}{2}(10 + 8) = 9$, so the area of APC is $\frac{1}{2} \cdot 21 \cdot 9 = \frac{189}{2}$.

5. Given that a, b, c are integers with $abc = 60$, and that complex number $\omega \neq 1$ satisfies $\omega^3 = 1$, find the minimum possible value of $\left|a + b\omega + c\omega^2\right|$.

   Proposed by: Ashwin Sah

   Answer: $\boxed{\sqrt{3}}$

   Since $\omega^3 = 1$, and $\omega \neq 1$, ω is a third root of unity. For any complex number z, $|z|^2 = z \cdot \bar{z}$. Letting $z = a + b\omega + c\omega^2$, we find that $\bar{z} = a + c\omega + b\omega^2$, and

$$\begin{aligned} |z|^2 &= a^2 + ab\omega + ac\omega^2 + ab\omega^2 + b^2 + bc\omega + ac\omega + bc\omega^2 + c^2 \\ &= (a^2 + b^2 + c^2) + (ab + bc + ca)(\omega) + (ab + bc + ca)(\omega^2) \\ &= (a^2 + b^2 + c^2) - (ab + bc + ca) \\ &= \frac{1}{2}((a-b)^2 + (b-c)^2 + (c-a)^2), \end{aligned}$$

   where we have used the fact that $\omega^3 = 1$ and that $\omega + \omega^2 = -1$. This quantity is minimized when a, b, and c are as close to each other as possible, making $a = 3, b = 4, c = 5$ the optimal choice, giving $|z|^2 = 3$. (A smaller value of $|z|$ requires two of a, b, c to be equal and the third differing from them by at most 2, which is impossible.) So $|z|_{\min} = \sqrt{3}$.

6. A positive integer n is *magical* if

$$\left\lfloor \sqrt{\lceil \sqrt{n} \rceil} \right\rfloor = \left\lceil \sqrt{\lfloor \sqrt{n} \rfloor} \right\rceil,$$

   where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ represent the floor and ceiling function respectively. Find the number of magical integers between 1 and 10,000, inclusive.

   Proposed by: Yuan Yao

   Answer: $\boxed{1330}$

   First of all, we have $\lfloor \sqrt{n} \rfloor = \lceil \sqrt{n} \rceil$ when n is a perfect square and $\lfloor \sqrt{n} \rfloor = \lceil \sqrt{n} \rceil - 1$ otherwise. Therefore, in the first case, the original equation holds if and only if $\sqrt{n}$ is a perfect square itself, i.e., n is a fourth power. In the second case, we need $m = \lfloor \sqrt{n} \rfloor$ to satisfy the equation $\lfloor \sqrt{m+1} \rfloor = \lceil \sqrt{m} \rceil$, which happens if and only if either m or $m+1$ is a perfect square k^2. Therefore, n is magical if and only if $(k^2-1)^2 < n < (k^2+1)^2$ for some (positive) integer k. There are $(k^2+1)^2 - (k^2-1)^2 = 4k^2 - 1$ integers in this range. The range in the problem statement includes $k = 1, 2, \ldots, 9$ and the interval $(99^2, 100^2]$, so the total number of magical numbers is

$$4(1^2 + 2^2 + \cdots + 9^2) - 9 + (100^2 - 99^2) = 4 \cdot \frac{9 \cdot (9+1) \cdot (18+1)}{6} + 190 = 1330.$$

7. Reimu has a wooden cube. In each step, she creates a new polyhedron from the previous one by cutting off a pyramid from each vertex of the polyhedron along a plane through the trisection point on each adjacent edge that is closer to the vertex. For example, the polyhedron after the first step has six octagonal faces and eight equilateral triangular faces. How many faces are on the polyhedron after the fifth step?

   Proposed by: Qi Qi

   Answer: $\boxed{974}$

   Notice that the number of vertices and edges triple with each step. We always have 3 edges meeting at one vertex, and slicing off a pyramid doesn't change this (we make new vertices from which one edge from the previous step and two of the pyramid edges emanate). So at each step we replace the sliced-off vertex with three new vertices, and to each edge we create four new "half-edges" (two from the pyramid at each endpoint), which is equivalent to tripling the number of vertices and edges. Then, by Euler's Theorem the number of faces is $E - V + 2 = 12 \cdot 3^5 - 8 \cdot 3^5 + 2 = 974$.

8. Marisa has a collection of $2^8 - 1 = 255$ distinct nonempty subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$. For each step she takes two subsets chosen uniformly at random from the collection, and replaces them with either their union or their intersection, chosen randomly with equal probability. (The collection is allowed to contain repeated sets.) She repeats this process $2^8 - 2 = 254$ times until there is only one set left in the collection. What is the expected size of this set?

   Proposed by: Yuan Yao

   Answer: $\boxed{\frac{1024}{255}}$

   It suffices to compute the probability of each number appearing in the final subset. For any given integer $n \in [1, 8]$, there are $2^7 = 128$ subsets with n and $2^7 - 1 = 127$ without. When we focus on only this element, each operation is equivalent to taking two random sets and discarding one of them randomly. Therefore there is a $\frac{128}{255}$ probability that n is in the final subset, and the expected value of its size is $8 \cdot \frac{128}{255} = \frac{1024}{255}$.

   Alternatively, since $|A|+|B| = |A \cup B|+|A \cap B|$, the expected value of the *average* size of all remaining subsets at a given step is constant, so the answer is simply the average size of all 255 subsets, which is $\frac{8 \cdot 128}{255} = \frac{1024}{255}$.

9. Find the minimum possible value of

$$\sqrt{58 - 42x} + \sqrt{149 - 140\sqrt{1 - x^2}}$$

   where $-1 \leq x \leq 1$.

   Proposed by: Serina Hu

   Answer: $\boxed{\sqrt{109}}$

   Substitute $x = \cos\theta$ and $\sqrt{1 - x^2} = \sin\theta$, and notice that $58 = 3^2 + 7^2$, $42 = 2 \cdot 3 \cdot 7$, $149 = 7^2 + 10^2$, and $140 = 2 \cdot 7 \cdot 10$. Therefore the first term is an application of Law of Cosines on a triangle that has two sides 3 and 7 with an angle measuring θ between them to find the length of the third side; similarly, the second is for a triangle with two sides 7 and 10 that have an angle measuring $90 - \theta$ between them. "Gluing" these two triangles together along their sides of length 7 so that the merged triangles form a right angle, we see that the minimum length of the sum of their third sides occurs when the glued triangles form a right triangle. This right triangle has legs of length 3 and 10, so its hypotenuse has length $\sqrt{109}$.

10. Five equally skilled tennis players named Allen, Bob, Catheryn, David, and Evan play in a round robin tournament, such that each pair of people play exactly once, and there are no ties. In each of the ten games, the two players both have a 50% chance of winning, and the results of the games are independent. Compute the probability that there exist four distinct players P_1, P_2, P_3, P_4 such that P_i beats P_{i+1} for $i = 1, 2, 3, 4$. (We denote $P_5 = P_1$).

    Proposed by: Steven Hao

    Answer: $\boxed{\frac{49}{64}}$

    We make the following claim: if there is a 5-cycle (a directed cycle involving 5 players) in the tournament, then there is a 4-cycle.

    Proof: Assume that A beats B, B beats C, C beats D, D beats E and E beats A. If A beats C then A, C, D, E forms a 4-cycle, and similar if B beats D, C beats E, and so on. However, if all five reversed matches occur, then A, D, B, C is a 4-cycle.

    Therefore, if there are no 4-cycles, then there can be only 3-cycles or no cycles at all.

    Case 1: There is a 3-cycle. Assume that A beats B, B beats C, and C beats A. (There are $\binom{5}{3} = 10$ ways to choose the cycle and 2 ways to orient the cycle.) Then D either beats all three or is beaten by all three, because otherwise there exists two people X and Y in these three people such that X beats Y, and D beats Y but is beaten by X, and then X, D, Y, Z will form a 4-cycle (Z is the remaining person of the three). The same goes for E. If D and E both beat all three or are beaten by all three, then there is no restriction on the match between D and E. However, if D beats all three and E loses to all three, then E cannot beat D because otherwise E, D, A, B forms a 4-cycle. This means that A, B, C is the only 3-cycle in the tournament, and once the cycle is chosen there are $2 \cdot 2 + 2 \cdot 1 = 6$ ways to choose the results of remaining matches, for $10 \cdot 2 \cdot 6 = 120$ ways in total.

Case 2: There are no cycles. This means that the tournament is a complete ordering (the person with a higher rank always beats the person with a lower rank). There are $5! = 120$ ways in this case as well.

Therefore, the probability of *not* having a 4-cycle is $\frac{120+120}{2^{10}} = \frac{15}{64}$, and thus the answer is $1 - \frac{15}{64} = \frac{49}{64}$.