HMMT November 2017, November 11, 2017 — GUTS ROUND

Organization ______________________________ Team ______________________________ Team ID# ___________

1. [**2**] Suppose x is a rational number such that $x\sqrt{2}$ is also rational. Find x.

2. [**2**] Regular octagon $CHILDREN$ has area 1. Find the area of pentagon $CHILD$.

3. [**2**] The length of a rectangle is three times its width. Given that its perimeter and area are both numerically equal to $k > 0$, find k.

HMMT November 2017, November 11, 2017 — GUTS ROUND

Organization ______________________________ Team ______________________________ Team ID# ___________

4. [**3**] Alec wishes to construct a string of 6 letters using the letters A, C, G, and N, such that:

   - The first three letters are pairwise distinct, and so are the last three letters;
   - The first, second, fourth, and fifth letters are pairwise distinct.

   In how many ways can he construct the string?

5. [**3**] Define a sequence $\{a_n\}$ by $a_1 = 1$ and $a_n = (a_{n-1})! + 1$ for every $n > 1$. Find the least n for which $a_n > 10^{10}$.

6. [**3**] Lunasa, Merlin, and Lyrica each have a distinct hat. Every day, two of these three people, selected randomly, switch their hats. What is the probability that, after 2017 days, every person has their own hat back?

HMMT November 2017, November 11, 2017 — GUTS ROUND

Organization ______________________________ Team ______________________________ Team ID# ___________

7. [**3**] Compute

$$100^2 + 99^2 - 98^2 - 97^2 + 96^2 + 95^2 - 94^2 - 93^2 + \ldots + 4^2 + 3^2 - 2^2 - 1^2.$$

8. [**3**] Regular octagon $CHILDREN$ has area 1. Find the area of quadrilateral $LINE$.

9. [**3**] A malfunctioning digital clock shows the time 9 : 57 AM; however, the correct time is 10 : 10 AM. There are two buttons on the clock, one of which increases the time displayed by 9 minutes, and another which decreases the time by 20 minutes. What is the minimum number of button presses necessary to correctly set the clock to the correct time?

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Organization ______________________________ Team ______________________________ Team ID# ___________

10. [**4**] Compute $\frac{x}{w}$ if $w \neq 0$ and $\frac{x+6y-3z}{-3x+4w} = \frac{-2y+z}{x-w} = \frac{2}{3}$.

11. [**4**] Find the sum of all real numbers x for which

$$\lfloor \lfloor \cdots \lfloor \lfloor \lfloor x \rfloor + x \rfloor + x \rfloor \cdots \rfloor + x \rfloor = 2017 \text{ and } \{\{\cdots\{\{\{x\} + x\} + x\} \cdots\} + x\} = \frac{1}{2017}$$

where there are 2017 x's in both equations. ($\lfloor x \rfloor$ is the integer part of x, and $\{x\}$ is the fractional part of x.) Express your sum as a mixed number.

12. [**4**] Trapezoid $ABCD$, with bases AB and CD, has side lengths $AB = 28$, $BC = 13$, $CD = 14$, and $DA = 15$. Let diagonals AC and BD intersect at P, and let E and F be the midpoints of AP and BP, respectively. Find the area of quadrilateral $CDEF$.

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Organization ______________________________ Team ______________________________ Team ID# ___________

13. [**5**] Fisica and Ritmo discovered a piece of Notalium shaped like a rectangular box, and wanted to find its volume. To do so, Fisica measured its three dimensions using a ruler with infinite precision, multiplied the results and rounded the product to the nearest cubic centimeter, getting a result of V cubic centimeters. Ritmo, on the other hand, measured each dimension to the nearest centimeter and multiplied the rounded measurements, getting a result of 2017 cubic centimeters. Find the positive difference between the least and greatest possible positive values for V.

14. [**5**] Points A, B, C, and D lie on a line in that order such that $\frac{AB}{BC} = \frac{DA}{CD}$. If $AC = 3$ and $BD = 4$, find AD.

15. [**5**] On a 3×3 chessboard, each square contains a knight with $\frac{1}{2}$ probability. What is the probability that there are two knights that can attack each other? (In chess, a knight can attack any piece which is two squares away from it in a particular direction and one square away in a perpendicular direction.)

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Organization ______________________________ Team ______________________________ Team ID# ___________

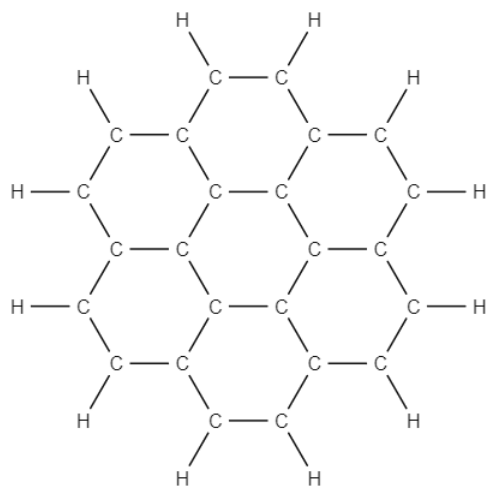
16. [**7**] A *repunit* is a positive integer, all of whose digits are 1s. Let $a_1 < a_2 < a_3 < \dots$ be a list of all the positive integers that can be expressed as the sum of distinct repunits. Compute a_{111}.

17. [**7**] A standard deck of 54 playing cards (with four cards of each of thirteen ranks, as well as two Jokers) is shuffled randomly. Cards are drawn one at a time until the first queen is reached. What is the probability that the next card is also a queen?

18. [**7**] Mr. Taf takes his 12 students on a road trip. Since it takes two hours to walk from the school to the destination, he plans to use his car to expedite the journey. His car can take at most 4 students at a time, and travels 15 times as fast as traveling on foot. If they plan their trip optimally, what is the shortest amount of time it takes for them to all reach the destination, in minutes?

Organization ______________________________ Team ______________________________ Team ID# ___________

19. [**9**] The skeletal structure of coronene, a hydrocarbon with the chemical formula $C_{24}H_{12}$, is shown below.



Each line segment between two atoms is at least a single bond. However, since each carbon (C) requires exactly four bonds connected to it and each hydrogen (H) requires exactly one bond, some of the line segments are actually double bonds. How many arrangements of single/double bonds are there such that the above requirements are satisfied? (Rotations and reflections of the same arrangement are considered distinct.)

20. [**9**] Rebecca has four resistors, each with resistance 1 ohm. Every minute, she chooses any two resistors with resistance of a and b ohms respectively, and combine them into one by one of the following methods:

- Connect them in series, which produces a resistor with resistance of $a + b$ ohms;
- Connect them in parallel, which produces a resistor with resistance of $\frac{ab}{a+b}$ ohms;
- Short-circuit one of the two resistors, which produces a resistor with resistance of either a or b ohms.

Suppose that after three minutes, Rebecca has a single resistor with resistance R ohms. How many possible values are there for R?

21. [**9**] A box contains three balls, each of a different color. Every minute, Randall randomly draws a ball from the box, notes its color, and then *returns it to the box.* Consider the following two conditions:

(1) Some ball has been drawn at least three times (not necessarily consecutively).

(2) Every ball has been drawn at least once.

What is the probability that condition (1) is met *before* condition (2)?

HMMT November 2017, November 11, 2017 — GUTS ROUND

Organization ________________________ Team ________________________ Team ID# ________

22. [**12**] A sequence of positive integers $a_1, a_2, \ldots, a_{2017}$ has the property that for all integers m where $1 \leq m \leq 2017$, $3(\sum_{i=1}^{m} a_i)^2 = \sum_{i=1}^{m} a_i^3$. Compute a_{1337}.

23. [**12**] A string of digits is defined to be *similar* to another string of digits if it can be obtained by reversing some contiguous substring of the original string. For example, the strings 101 and 110 are similar, but the strings 3443 and 4334 are not. (Note that a string is always similar to itself.) Consider the string of digits

$$S = 01234567890123456789012345678901234567890123456789,$$

consisting of the digits from 0 to 9 repeated five times. How many distinct strings are similar to S?

24. [**12**] Triangle ABC has side lengths $AB = 15, BC = 18, CA = 20$. Extend CA and CB to points D and E respectively such that $DA = AB = BE$. Line AB intersects the circumcircle of CDE at P and Q. Find the length of PQ.

HMMT November 2017, November 11, 2017 — GUTS ROUND

Organization ________________________ Team ________________________ Team ID# ________

Déjà vu?

25. [**15**] Fisica and Ritmo discovered a piece of Notalium shaped like a rectangular box, and wanted to find its volume. To do so, Fisica measured its three dimensions using a ruler with infinite precision, multiplied the results and rounded the product to the nearest cubic centimeter, getting a result of 2017 cubic centimeters. Ritmo, on the other hand, measured each dimension to the nearest centimeter and multiplied the rounded measurements, getting a result of V cubic centimeters. Find the positive difference between the least and greatest possible positive values for V.

26. [**15**] Points A, B, C, D lie on a circle in that order such that $\dfrac{AB}{BC} = \dfrac{DA}{CD}$. If $AC = 3$ and $BD = BC = 4$, find AD.

27. [**15**] On a 3×3 chessboard, each square contains a Chinese knight with $\frac{1}{2}$ probability. What is the probability that there are two Chinese knights that can attack each other? (In Chinese chess, a Chinese knight can attack any piece which is two squares away from it in a particular direction and one square away in a perpendicular direction, under the condition that there is no other piece immediately adjacent to it in the first direction.)

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Organization ______________________________ Team ______________________________ Team ID# ___________

28. [**19**] Compute the number of functions $f : \{1, 2, \ldots, 9\} \to \{1, 2, \ldots, 9\}$ which satisfy $f(f(f(f(f(x))))) = x$ for each $x \in \{1, 2, \ldots, 9\}$.

29. [**19**] Consider a sequence x_n such that $x_1 = x_2 = 1$, $x_3 = \frac{2}{3}$. Suppose that $x_n = \frac{x_{n-1}^2 x_{n-2}}{2x_{n-2}^2 - x_{n-1}x_{n-3}}$ for all $n \geq 4$. Find the least n such that $x_n \leq \frac{1}{10^6}$.

30. [**19**] Given complex number z, define sequence $z_0, z_1, z_2, \ldots$ as $z_0 = z$ and $z_{n+1} = 2z_n^2 + 2z_n$ for $n \geq 0$. Given that $z_{10} = 2017$, find the minimum possible value of $|z|$.

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HMMT November 2017, November 11, 2017 — GUTS ROUND

Organization ______________________________ Team ______________________________ Team ID# ___________

31. [**24**] In unit square $ABCD$, points E, F, G are chosen on side BC, CD, DA respectively such that AE is perpendicular to EF and EF is perpendicular to FG. Given that $GA = \frac{404}{1331}$, find all possible values of the length of BE.

32. [**24**] Let P be a polynomial with integer coefficients such that $P(0) + P(90) = 2018$. Find the least possible value for $|P(20) + P(70)|$.

33. [**24**] Tetrahedron $ABCD$ with volume 1 is inscribed in circumsphere ω such that $AB = AC = AD = 2$ and $BC \cdot CD \cdot DB = 16$. Find the radius of ω.

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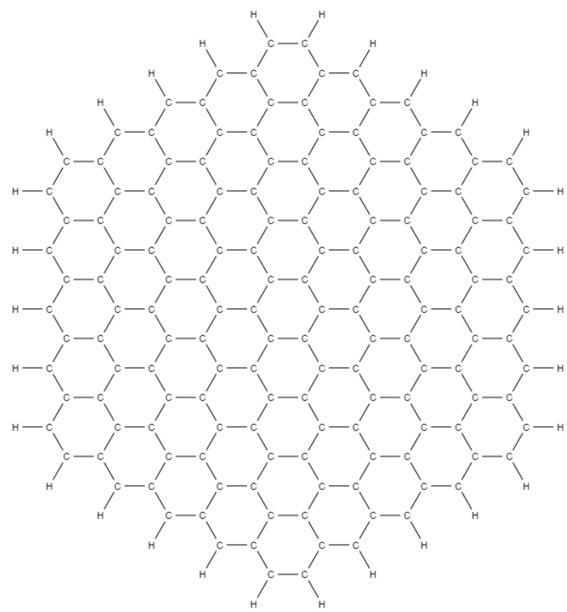
Organization ____________________________ Team ____________________________ Team ID# __________

Déjà vu??

34. [**30**] The skeletal structure of circumcircumcircumcoronene, a hydrocarbon with the chemical formula $C_{150}H_{30}$, is shown below.



Each line segment between two atoms is at least a single bond. However, since each carbon (C) requires exactly four bonds connected to it and each hydrogen (H) requires exactly one bond, some of the line segments are actually double bonds. How many arrangements of single/double bonds are there such that the above requirements are satisfied? (Rotations and reflections of the same arrangement are considered distinct.)

If the correct answer is C and your answer is A, you get $\max\left(\left\lfloor 30\left(1-\left|\log_{\log_2 C}\frac{A}{C}\right|\right)\right\rfloor, 0\right)$ points.

35. [**30**] Rebecca has twenty-four resistors, each with resistance 1 ohm. Every minute, she chooses any two resistors with resistance of a and b ohms respectively, and combine them into one by one of the following methods:

- Connect them in series, which produces a resistor with resistance of $a+b$ ohms;
- Connect them in parallel, which produces a resistor with resistance of $\frac{ab}{a+b}$ ohms;
- Short-circuit one of the two resistors, which produces a resistor with resistance of either a or b ohms.

Suppose that after twenty-three minutes, Rebecca has a single resistor with resistance R ohms. How many possible values are there for R?

If the correct answer is C and your answer is A, you get $\max\left(\left\lfloor 30\left(1-\left|\log_{\log_2 C}\frac{A}{C}\right|\right)\right\rfloor, 0\right)$ points.

36. [**30**] A box contains twelve balls, each of a different color. Every minute, Randall randomly draws a ball from the box, notes its color, and then *returns it to the box.* Consider the following two conditions:

(1) Some ball has been drawn at least twelve times (not necessarily consecutively).

(2) Every ball has been drawn at least once.

What is the probability that condition (1) is met *before* condition (2)?

If the correct answer is C and your answer is A, you get $\max\left(\left\lfloor 30\left(1-\frac{1}{2}\left|\log_2 A-\log_2 C\right|\right)\right\rfloor, 0\right)$ points.