HMMT November 2017

November 11, 2017

Guts Round

1. [**2**] Suppose x is a rational number such that $x\sqrt{2}$ is also rational. Find x.

   Proposed by: Michael Kural

   Answer: $\boxed{0}$

   Suppose $x \neq 0$. Then $\frac{x\sqrt{2}}{x} = \sqrt{2}$ is the quotient of two nonzero rationals, and so is rational. However, it is well-known that $\sqrt{2}$ is irrational. Therefore any solution x must satisfy $x = 0$. We can see that 0 is rational and $0\sqrt{2} = 0$ is rational, so the answer is indeed $x = \boxed{0}$.

2. [**2**] Regular octagon $CHILDREN$ has area 1. Find the area of pentagon $CHILD$.

   Proposed by: Yuan Yao

   Answer: $\boxed{\frac{1}{2}}$

   The pentagon $CHILD$ is congruent to the pentagon $NERDC$, as their corresponding angles and sides are congruent. Moreover, the two pentagons together compose the entire octagon, so each pentagon must have area one-half of the area of the octagon, or $\frac{1}{2}$.

3. [**2**] The length of a rectangle is three times its width. Given that its perimeter and area are both numerically equal to $k > 0$, find k.

   Proposed by: Yuan Yao

   Answer: $\boxed{\frac{64}{3}}$

   Let a be the width of the rectangle. Then the length of the rectangle is $3a$, so the perimeter is $2(a + 3a) = 8a$, and the area is $3a^2$. Since the length is numerically equal to the width, we know that

   $$8a = 3a^2 = k.$$

   Because $k > 0$, the rectangle is non-degenerate. It follows that $8 = 3a$, so $a = \frac{8}{3}$. Therefore, $k = \boxed{\dfrac{64}{3}}$.

4. [**3**] Alec wishes to construct a string of 6 letters using the letters A, C, G, and N, such that:

   - The first three letters are pairwise distinct, and so are the last three letters;
   - The first, second, fourth, and fifth letters are pairwise distinct.

   In how many ways can he construct the string?

   Proposed by: Yuan Yao

   There are $4! = 24$ ways to decide the first, second, fourth, and fifth letters because these letters can be selected sequentially without replacement from the four possible letters. Once these four letters are selected, there are 2 ways to select the third letter because two distinct letters have already been selected for the first and second letters, leaving two possibilities. The same analysis applies to the sixth letter. Thus, there are $24 \cdot 2^2 = \boxed{96}$ total ways to construct the string.

5. [**3**] Define a sequence $\{a_n\}$ by $a_1 = 1$ and $a_n = (a_{n-1})! + 1$ for every $n > 1$. Find the least n for which $a_n > 10^{10}$.

   Proposed by: Michael Tang

   Answer: $\boxed{6}$

   We have $a_2 = 2$, $a_3 = 3$, $a_4 = 7$, $a_5 = 7! + 1 = 5041$, and $a_6 = 5041! + 1$. But

   $$5041! + 1 \gg 5041 \cdot 5040 \cdot 5039 > 10^{10}.$$

   Hence, the answer is $\boxed{6}$.

6. [**3**] Lunasa, Merlin, and Lyrica each have a distinct hat. Every day, two of these three people, selected randomly, switch their hats. What is the probability that, after 2017 days, every person has their own hat back?

Proposed by: Yuan Yao

Answer: $\boxed{0}$

Imagine that the three hats are the vertices of an equilateral triangle. Then each day the exchange is equivalent to reflecting the triangle along one of its three symmetry axes, which changes the orientation of the triangle (from clockwise to counterclockwise or vice versa). Thus, an even number of such exchanges must be performed if the orientation is to be preserved. Since the triangle is reflected 2017 times, it is impossible for the final triangle to have the same orientation as the original triangle, so the desired probability is $\boxed{0}$.

7. [**3**] Compute

$$100^2 + 99^2 - 98^2 - 97^2 + 96^2 + 95^2 - 94^2 - 93^2 + \ldots + 4^2 + 3^2 - 2^2 - 1^2.$$

Proposed by: Michael Tang

Answer: $\boxed{10100}$

Note that $(n+3)^2 - (n+2)^2 - (n+1)^2 + n^2 = 4$ for every n. Therefore, adding 0^2 to the end of the given sum and applying this identity for every four consecutive terms after 100^2, we see that the given sum is equivalent to $100^2 + 25 \cdot 4 = \boxed{10100}$.

Alternatively, we can apply the difference-of-squares factorization to rewrite $100^2 - 98^2 = (100 - 98)(100+98) = 2(100+98)$, $99^2 - 97^2 = (99-97)(99+97) = 2(99+97)$, etc. Thus, the given sum is equivalent to $2(100 + 99 + \cdots + 2 + 1) = 2 \cdot \frac{100 \cdot 101}{2} = 10100$.

8. [**3**] Regular octagon $CHILDREN$ has area 1. Determine the area of quadrilateral $LINE$.

Proposed by: Yuan Yao

Answer: $\boxed{\frac{1}{2}}$

Suppose that the side length $CH = \sqrt{2}a$, then the area of the octagon is $((2+\sqrt{2})a)^2 - 4 \cdot \frac{1}{2}a^2 = (4+4\sqrt{2})a^2$, and the area of $LINE$ is $(\sqrt{2}a)((2+\sqrt{2})a) = (2+2\sqrt{2})a^2$, which is exactly one-half of the area of the octagon. Therefore the area of $LINE$ is $\frac{1}{2}$.

9. [**3**] A malfunctioning digital clock shows the time $9:57$ AM; however, the correct time is $10:10$ AM. There are two buttons on the clock, one of which increases the time displayed by 9 minutes, and another which decreases the time by 20 minutes. What is the minimum number of button presses necessary to correctly set the clock to the correct time?

Proposed by: Kevin Sun

Answer: $\boxed{24}$

We need to increase the time by 13 minutes. If we click the 9 minute button a times and the 20 minute button b times, then we must have $9a - 20b = 13$. Note that if this equation is satisfied, then b increases as a increases, so it suffices to minimize a. This means that a must end in a 7. However, since $63 - 20b = 13$ has no integer solution, the next smallest possible value of a is 17, which gives the solution $(a, b) = (17, 7)$, or 24 button presses.

10. [**4**] Compute $\dfrac{x}{w}$ if $w \neq 0$ and $\dfrac{x + 6y - 3z}{-3x + 4w} = \dfrac{-2y + z}{x - w} = \dfrac{2}{3}$.

Proposed by: Angela Deng

Answer: $\boxed{\frac{2}{3}}$

We have $x + 6y - 3z = \frac{2}{3}(-3x + 4w)$ and $-2y + z = \frac{2}{3}(x - w)$, so

$$\frac{x}{w} = \frac{(x+6y-3z)+3(-2y+z)}{(-3x+4w)+3(x-w)} = \frac{\frac{2}{3}(-3x+4w)+3\cdot\frac{2}{3}(x-w)}{(-3x+4w)+3(x-w)} = \frac{\frac{2}{3}[(-3x+4w)+3(x-w)]}{(-3x+4w)+3(x-w)} = \boxed{\frac{2}{3}}$$

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11. [**4**] Find the sum of all real numbers x for which

$$\lfloor\lfloor\cdots\lfloor\lfloor\lfloor x\rfloor + x\rfloor + x\rfloor\cdots\rfloor + x\rfloor = 2017 \text{ and } \{\{\cdots\{\{\{x\} + x\} + x\}\cdots\} + x\} = \frac{1}{2017}$$

where there are 2017 x's in both equations. ($\lfloor x\rfloor$ is the integer part of x, and $\{x\}$ is the fractional part of x.) Express your sum as a mixed number.

Proposed by: Yuan Yao

Answer: $\boxed{3025\frac{1}{2017} \text{ or } \frac{6101426}{2017}}$

The two equations are equivalent to $2017\lfloor x\rfloor = 2017$ and $\{2017x\} = \frac{1}{2017}$, respectively. The first equation reduces to $\lfloor x\rfloor = 1$, so we must have $x = 1 + r$ for some real r satisfying $0 \le r < 1$. From the second equation, we deduce that $\{2017x\} = \{2017 + 2017r\} = \{2017r\} = \frac{1}{2017}$, so $2017r = n + \frac{1}{2017}$, where n is an integer. Dividing both sides of this equation by 2017 yields $r = \frac{n}{2017} + \frac{1}{2017^2}$, where $n = 0, 1, 2, \ldots, 2016$ so that we have $0 \le r < 1$. Thus, we have $x = 1 + r = 1 + \frac{n}{2017} + \frac{1}{2017^2}$ for $n = 0, 1, 2, \ldots, 2016$. The sum of these solutions is $2017 \cdot 1 + \frac{2016\cdot 2017}{2} \cdot \frac{1}{2017} + 2017 \cdot \frac{1}{2017^2} = 2017 + \frac{2016}{2} + \frac{1}{2017} = \boxed{3025\frac{1}{2017}}$.

12. [**4**] Trapezoid $ABCD$, with bases AB and CD, has side lengths $AB = 28$, $BC = 13$, $CD = 14$, and $DA = 15$. Let diagonals AC and BD intersect at P, and let E and F be the midpoints of AP and BP, respectively. Find the area of quadrilateral $CDEF$.

Proposed by: Christopher Shao

Answer: $\boxed{112}$

Note that EF is a midline of triangle APB, so EF is parallel to AB and $EF = \frac{1}{2}AB = 14 = CD$. We also have that EF is parallel to CD, and so $CDEF$ is a parallelogram. From this, we have $EP = PC$ as well, so $\frac{CE}{CA} = \frac{2}{3}$. It follows that the height from C to EF is $\frac{2}{3}$ of the height from C to AB. We can calculate that the height from C to AB is 12, so the height from C to EF is 8. Therefore $CDEF$ is a parallelogram with base 14 and height 8, and its area is $14 \cdot 8 = 112$.

13. [**5**] Fisica and Ritmo discovered a piece of Notalium shaped like a rectangular box, and wanted to find its volume. To do so, Fisica measured its three dimensions using a ruler with infinite precision, multiplied the results and rounded the product to the nearest cubic centimeter, getting a result of V cubic centimeters. Ritmo, on the other hand, measured each dimension to the nearest centimeter and multiplied the rounded measurements, getting a result of 2017 cubic centimeters. Find the positive difference between the least and greatest possible positive values for V.

Proposed by: Yuan Yao

Answer: $\boxed{4035}$

The only possible way for Ritmo to get 2017 cubic centimeters is to have his measurements rounded to $1, 1, 2017$ centimeters respectively. Therefore the largest value of V is achieved when the dimensions are $(1.5 - \epsilon)(1.5 - \epsilon)(2017.5 - \epsilon) = 4539.375 - \epsilon'$ for some very small positive real ϵ, ϵ', and the smallest value of V is achieved when the dimensions are $(0.5 + \epsilon)(0.5 + \epsilon)(2016.5 + \epsilon) = 504.125 + \epsilon'$ for some very small positive real ϵ, ϵ'. Therefore the positive difference is $4539 - 504 = 4035$.

14. [**5**] Points A, B, C, and D lie on a line in that order such that $\dfrac{AB}{BC} = \dfrac{DA}{CD}$. If $AC = 3$ and $BD = 4$, find AD.

Proposed by: Yuan Yao

Answer: $\boxed{6}$

Let $BC = x$, then the equation becomes $\frac{3-x}{x} = \frac{7-x}{4-x}$. This simplifies to a quadratic equation with solutions $x = 1$ and $x = 6$. Since $x < 3$, we have $x = 1$ and $AD = 7 - x = 6$.

15. [**5**] On a 3×3 chessboard, each square contains a knight with $\frac{1}{2}$ probability. What is the probability that there are two knights that can attack each other? (In chess, a knight can attack any piece which is two squares away from it in a particular direction and one square away in a perpendicular direction.)

Proposed by: Yuan Yao

Answer: $\boxed{\frac{209}{256}}$

Notice that a knight on the center square cannot attack any other square on the chessboard, so whether it contains a knight or not is irrelevant.

For ease of reference, we label the other eight squares as follows:

0	5	2
3	X	7
6	1	4

Notice that a knight in square i attacks both square $i+1$ and $i-1$ (where square numbers are reduced modulo 8). We now consider the number of ways such that *no* two knights attack each other.

- 0 knights: 1 way.
- 1 knights: 8 ways.
- 2 knights: $\binom{8}{2} - 8 = 20$ ways.
- 3 knights: $8 + 8 = 16$ ways, where the two 8s represent the number of ways such that the "distances" between the knights (index-wise) are $2, 2, 4$ and $2, 3, 3$ respectively.
- 4 knights: 2 ways.

Therefore, out of $2^8 = 256$ ways, $1 + 8 + 20 + 16 + 2 = 47$ of them doesn't have a pair of attacking knights. Thus the answer is $\frac{256-47}{256} = \frac{209}{256}$.

16. [**7**] A *repunit* is a positive integer, all of whose digits are 1s. Let $a_1 < a_2 < a_3 < \dots$ be a list of all the positive integers that can be expressed as the sum of distinct repunits. Compute a_{111}.

Proposed by: Michael Tang

Answer: $\boxed{1223456}$

Let $\{r_n\}_{n \geq 0}$ be the repunits (so $r_0 = 1$, $r_1 = 11$, and so on). We see that for any n, there is

$$r_{n-1} + r_{n-2} + \cdots + r_0 < \frac{r_n}{10} + \frac{r_n}{100} + \cdots < \frac{r_n}{9} < r_n,$$

so r_n is only needed when all possible combinations of the first n repunits are exhausted (after 2^n terms), which shows that there is a bijection between the sequence $\{a_n\}$ and the binary numbers. In particular, if $k = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_s}$ for distinct n_i's, then $a_k = r_{n_1} + r_{n_2} + \cdots + r_{n_s}$. Since $111 = 1101111_2 = 2^0 + 2^1 + 2^2 + 2^3 + 2^5 + 2^6$, we have

$$a_{111} = r_0 + r_1 + r_2 + r_3 + r_5 + r_6 = 1223456.$$

17. [**7**] A standard deck of 54 playing cards (with four cards of each of thirteen ranks, as well as two Jokers) is shuffled randomly. Cards are drawn one at a time until the first queen is reached. What is the probability that the next card is also a queen?

Proposed by: Serina Hu

Answer: $\boxed{\frac{2}{27}}$

Since the four queens are equivalent, we can compute the probability that a specific queen, say the queen of hearts, is right after the first queen. Remove the queen of hearts; then for every ordering of the 53 other cards, there are 54 locations for the queen of hearts, and exactly one of those is after the first queen. Therefore the probability that the queen of hearts immediately follows the first queen is $\frac{1}{54}$, and the probability any queen follows the first queen is $\frac{1}{54} \cdot 4 = \frac{2}{27}$.

18. [**7**] Mr. Taf takes his 12 students on a road trip. Since it takes two hours to walk from the school to the destination, he plans to use his car to expedite the journey. His car can take at most 4 students at a time, and travels 15 times as fast as traveling on foot. If they plan their trip optimally, what is the shortest amount of time it takes for them to all reach the destination, in minutes?

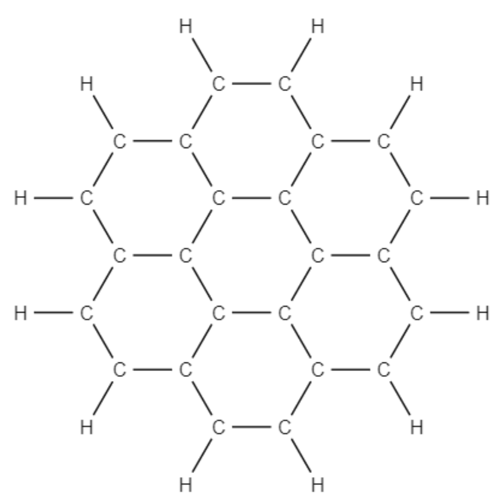
Proposed by: Yuan Yao

Answer: $\boxed{30.4 \text{ OR } \frac{152}{5}}$

A way to plan the trip is to have Mr. Taf drive 4 students to the 80% mark, then drive back to the 10% mark to pick up another 4 students to the 90% mark, and finally drive back to the 20% mark to pick up the last 4 students to the destination. All students will reach the destination at the same time, and Mr. Taf would have driven for $(0.8 + 0.7 + 0.8 + 0.7 + 0.8) \cdot \frac{120}{15} = 30.4$ minutes.

Now we show that 30.4 minutes is necessary. First of all, for a trip to be optimal, Mr. Taf must not carry students when he was driving away from the destination, and all student not on the car must keep walking towards the destination. Suppose that among all the students, the student that walked for the longest walked for $15m$ minutes, where $0 \leq m \leq 8$, then he spent $\frac{120-15m}{15} = 8 - m$ minutes on the car, so it took them exactly $14m + 8$ minutes to get to the destination. Moreover, all students must have spent at least $12(8 - m)$ minutes on the car total, and Mr. Taf would need to spend at least $3(8 - m) = 24 - 3m$ minutes driving students *towards* the destination. Since it takes Mr. Taf 8 minutes to drive the entire trip, he would need to drive $3(8 - m) - 8 = 16 - 3m$ minutes *away* from the destination, so Mr. Fat drove for at least $40 - 6m$ minutes. From this we derive the inequality $40 - 6m \geq 14m + 8$, which comes out to $m \geq 1.6$, so the journey is at least $14(1.6) + 8 = 30.4$ minutes.

19. [**9**] The skeletal structure of coronene, a hydrocarbon with the chemical formula $C_{24}H_{12}$, is shown below.



Each line segment between two atoms is at least a single bond. However, since each carbon (C) requires exactly four bonds connected to it and each hydrogen (H) requires exactly one bond, some of the line segments are actually double bonds. How many arrangements of single/double bonds are there such that the above requirements are satisfied?

Proposed by: Yuan Yao

Answer: $\boxed{20}$

Note that each carbon needs exactly one double bond. Label the six carbons in the center $1, 2, 3, 4, 5, 6$ clockwise. We consider how these six carbons are double-bonded. If a carbon in the center is not double-bonded to another carbon in the center, it must double-bond to the corresponding carbon on the outer ring. This will result in the outer ring broken up into (some number of) strings instead of a loop, which means that there will be at most one way to pair off the outer carbons through double-bonds. (In fact, as we will demonstrate later, there will be exactly one way.)

Now we consider how many double bonds are on the center ring.

- 3 bonds. There are 2 ways to pair of the six carbons, and 2 ways to pair of the outer ring as well, for 4 ways in total.
- 2 bonds. Then either two adjacent carbons (6 ways) or two diametrically opposite carbons (3 ways) are not double-bonded, and in the former case the outer ring will be broken up into two "strands" with 2 and 14 carbons each, while in the latter case it will be broken up into two strands both with 8 carbons each, and each produce one valid way of double-bonding, for 9 ways in total.
- 1 bond. There are 6 ways to choose the two double-bonded center carbon, and the outer ring will be broken up into four strands with 2, 2, 2, 8 carbons each, which gives one valid way of double-bonding, for 6 ways in total.
- 0 bonds. Then the outer ring is broken up into six strands of 2 carbons each, giving 1 way.

Therefore, the number of possible arrangements is $4 + 9 + 6 + 1 = 20$.

Note: each arrangement of single/double bonds is also called a *resonance structure* of coronene.

20. [**9**] Rebecca has four resistors, each with resistance 1 ohm. Every minute, she chooses any two resistors with resistance of a and b ohms respectively, and combine them into one by one of the following methods:

- Connect them in series, which produces a resistor with resistance of $a + b$ ohms;
- Connect them in parallel, which produces a resistor with resistance of $\frac{ab}{a+b}$ ohms;
- Short-circuit one of the two resistors, which produces a resistor with resistance of either a or b ohms.

Suppose that after three minutes, Rebecca has a single resistor with resistance R ohms. How many possible values are there for R?

Proposed by: Yuan Yao

Answer: $\boxed{15}$

Let R_n be the set of all possible resistances using exactly n 1-ohm circuit segments (without shorting any of them), then we get $R_n = \bigcup_{i=1}^{n-1} \left(\{a + b \mid a \in R_i, b \in R_{n-i}\} \cup \{\frac{ab}{a+b} \mid a \in R_i, b \in R_{n-i}\}\right)$, starting with $R_1 = \{1\}$, we get:

$$R_2 = \{\frac{1}{2}, 2\}$$
$$R_3 = \{\frac{1}{3}, \frac{2}{3}, \frac{3}{2}, 3\}$$
$$R_4 = \{\frac{1}{4}, \frac{2}{5}, \frac{3}{5}, \frac{3}{4}, 1, \frac{4}{3}, \frac{5}{3}, \frac{5}{2}, 4\}$$

Their union is the set of all possible effective resistances we can get, which contains $2 + 4 + 9 = 15$ values. (Note that $R_1 \subset R_4$ and the sets R_2, R_3, R_4 are disjoint.)

21. [**9**] A box contains three balls, each of a different color. Every minute, Randall randomly draws a ball from the box, notes its color, and then *returns it to the box.* Consider the following two conditions:

(1) Some ball has been drawn at least three times (not necessarily consecutively).

(2) Every ball has been drawn at least once.

What is the probability that condition (1) is met *before* condition (2)?

Proposed by: Yuan Yao

Answer: $\boxed{\frac{13}{27}}$

At any time, we describe the current state by the number of times each ball is drawn, sorted in nonincreasing order. For example, if the red ball has been drawn twice and green ball once, then the state would be $(2, 1, 0)$. Given state S, let P_S be the probability that the state was achieved at some point of time before one of the two conditions are satisfied. Starting with $P_{(0,0,0)} = 1$, we compute:

$$P_{(1,0,0)} = 1;$$

$$P_{(1,1,0)} = \frac{2}{3},\ P_{(2,0,0)} = \frac{1}{3};$$

$$P_{(1,1,1)} = \frac{1}{3}P_{(1,1,0)} = \frac{2}{9},\ P_{(2,1,0)} = \frac{2}{3}P_{(1,1,0)} + \frac{2}{3}P_{(2,0,0)} = \frac{2}{3},\ P_{(3,0,0)} = \frac{1}{3}P_{(2,0,0)} = \frac{1}{9};$$

$$P_{(2,1,1)} = P_{(2,2,0)} = P_{(3,1,0)} = \frac{1}{3}P_{(2,1,0)} = \frac{2}{9};$$

$$P_{(2,2,1)} = \frac{1}{3}P_{(2,2,0)} = \frac{2}{27},\ P_{(3,2,0)} = \frac{2}{3}P_{(2,2,0)} = \frac{4}{27}.$$

Therefore, the probability that the first condition is satisfied first is $P_{(3,0,0)} + P_{(3,1,0)} + P_{(3,2,0)} = \frac{1}{9} + \frac{2}{9} + \frac{4}{27} = \frac{13}{27}$.

22. [**12**] A sequence of positive integers $a_1, a_2, \ldots, a_{2017}$ has the property that for all integers m where $1 \le m \le 2017$, $3(\sum_{i=1}^{m} a_i)^2 = \sum_{i=1}^{m} a_i^3$. Compute a_{1337}.

Proposed by: Serina Hu

Answer: $\boxed{4011}$

I claim that $a_i = 3i$ for all i. We can conjecture that the sequence should just be the positive multiples of three because the natural numbers satisfy the property that the square of their sum is the sum of their cubes, and prove this by induction. At $i = 1$, we have that $3a_i^2 = a_i^3$, so $a_i = 3$. Now assuming this holds for $i = m$, we see that

$$\begin{aligned}
3\left(\sum_{i=1}^{m+1} a_i\right)^2 &= 3\left(a_{m+1} + \sum_{i=1}^{m} a_i\right)^2 \\
&= 3a_{m+1}^2 + \sum_{i=1}^{m} a_i^3 + 6a_{m+1}\sum_{i=1}^{m} a_i \\
&= 3a_{m+1}^2 + \sum_{i=1}^{m} a_i^3 + 6a_{m+1} \cdot 3\left(\frac{m(m+1)}{2}\right) \\
&= \sum_{i=1}^{m+1} a_i^3.
\end{aligned}$$

Therefore,

$$\begin{aligned}
a_{m+1}^3 &= 3a_{m+1}^2 + a_{m+1}(9m^2 + 9m) \\
0 &= a_{m+1}^2 - 3a_{m+1} - (9m^2 + 9m) \\
0 &= (a_{m+1} - (3m + 3))(a_{m+1} + 3m)
\end{aligned}$$

and because the sequence is positive, $a_{m+1} = 3m + 3$, which completes the induction. Then $a_{1337} = 1337 \cdot 3 = 4011$.

23. [**12**] A string of digits is defined to be *similar* to another string of digits if it can be obtained by reversing some contiguous substring of the original string. For example, the strings 101 and 110 are similar, but the strings 3443 and 4334 are not. (Note that a string is always similar to itself.) Consider the string of digits

$$S = 01234567890123456789012345678901234567890123456789,$$

consisting of the digits from 0 to 9 repeated five times. How many distinct strings are similar to S?

Proposed by: Kevin Sun

Answer: $\boxed{1126}$

We first count the number of substrings that one could pick to reverse to yield a new substring. If we insert two dividers into the sequence of 50 digits, each arrangement of 2 dividers among the 52 total objects specifies a substring that is contained between the two dividers, for a total of $\binom{52}{2}$ substrings. Next, we account for overcounting. Every substring of length 0 or 1 will give the identity string when reversed, so we are overcounting here by $51 + 50 - 1 = 100$ substrings. Next, for any longer substring s that starts and ends with the same digit, removing the digit from both ends results in a substring s', such that reversing s would give the same string as reversing s'. Therefore, we are overcounting by $10 \cdot \binom{5}{2}$ substrings. Our total number of strings similar to S is therefore $\binom{52}{2} - 100 - 10 \cdot \binom{5}{2} = \boxed{1126}$.

24. [**12**] Triangle ABC has side lengths $AB = 15, BC = 18, CA = 20$. Extend CA and CB to points D and E respectively such that $DA = AB = BE$. Line AB intersects the circumcircle of CDE at P and Q. Find the length of PQ.

Proposed by: Yuan Yao

Answer: $\boxed{37}$

WLOG suppose that P is closer to A than to B. Let $DA = AB = BE = c = 15, BC = a = 18, CA = b = 20, PA = x$, and $QB = y$. By Power of a Point on B and A, we get $ac = (x+c)y$ and $bc = (y+c)x$, respectively. Subtracting the two equations gives $cy - cx = ac - bc \Rightarrow y - x = a - b$. Substituting $y = x + a - b$ into the first equation gives $ac = (x + c)(x + a - b) = x^2 + (a - b + c)x + ac - bc$, which is a quadratic with unique positive solution $x = \frac{(b-a-c)+\sqrt{(a-b+c)^2+4bc}}{2}$. Thus,

$$PQ = x+y+c = (y-x)+2x+c = (a-b+c)+(b-a-c)+\sqrt{(a-b+c)^2+4bc} = \sqrt{13^2+4\cdot 20\cdot 15} = 37.$$

25. [**15**] Fisica and Ritmo discovered a piece of Notalium shaped like a rectangular box, and wanted to find its volume. To do so, Fisica measured its three dimensions using a ruler with infinite precision, multiplied the results and rounded the product to the nearest cubic centimeter, getting a result of 2017 cubic centimeters. Ritmo, on the other hand, measured each dimension to the nearest centimeter and multiplied the rounded measurements, getting a result of V cubic centimeters. Find the positive difference between the least and greatest possible positive values for V.

Proposed by: Yuan Yao

Answer: $\boxed{7174}$

It is not difficult to see that the maximum possible value of V can be achieved when the dimensions are $(0.5 + \epsilon) \times (0.5 + \epsilon) \times (8070 - \epsilon') = 2017.5 - \epsilon''$ for some very small reals $\epsilon, \epsilon', \epsilon'' > 0$, which when measured by Ritmo, gives $V = 1 \cdot 1 \cdot 8070 = 8070$. Similarly, the minimum possible *positive* value of V can be achieved when the dimensions are $(1.5 - \epsilon) \times (1.5 - \epsilon) \times (\frac{8066}{9} + \epsilon') = 2016.5 + \epsilon''$ for some very small reals $\epsilon, \epsilon', \epsilon'' > 0$, which when measured by Ritmo, gives $V = 1 \cdot 1 \cdot 896 = 896$. Therefore, the difference between the maximum and minimum is $8070 - 896 = 7174$.

26. [**15**] Points A, B, C, D lie on a circle in that order such that $\frac{AB}{BC} = \frac{DA}{CD}$. If $AC = 3$ and $BD = BC = 4$, find AD.

Proposed by: Yuan Yao

Answer: $\boxed{\frac{3}{2}}$

By Ptolemy's theorem, we have $AB \cdot CD + BC \cdot DA = AC \cdot BD = 3 \cdot 4 = 12$. Since the condition implies $AB \cdot CD = BC \cdot DA$, we have $DA = \frac{6}{BC} = \frac{3}{2}$.

27. [**15**] On a 3×3 chessboard, each square contains a Chinese knight with $\frac{1}{2}$ probability. What is the probability that there are two Chinese knights that can attack each other? (In Chinese chess, a Chinese knight can attack any piece which is two squares away from it in a particular direction and one square away in a perpendicular direction, under the condition that there is no other piece immediately adjacent to it in the first direction.)

Proposed by: Yuan Yao

Answer: $\boxed{\frac{79}{256}}$

Suppose the 3×3 square is

A	B	C
D	E	F
G	H	I

We count the number of ways a board could have two knights attack each other using PIE. First notice that in any setup with two knights attack each other, the center square must be empty. Also, for any pair of knights that attack each other, one must be in a corner, and the other at the center of a nonadjacent side. There are $8 \cdot 2^5$ ways for one pair of knights to attack each other. Next, we count the number of ways two pairs of knights attack each other: up to symmetry, there are four cases: knights at A, B, G, H, and D and E empty; knights at A, H, F, and B, D, E empty; knights at A, B, H, I, and D, E, F empty; and knights at A, C, H, and D, E, F empty. Four each of these cases, there are four symmetries, so there are a total of $4 \cdot (2^3 + 2^3 + 2^2 + 2^3)$ ways to have two pairs of knights attack each other. Next, there's only one way for three pairs of knights to attack each other, discounting symmetry: A, B, G, H, I have knights, and D, E, F empty. Then there are $4 \cdot 2 \cdot 2$ ways for three knights to attack. Finally, there is only one way for four knights to attack: knights at A, B, C, G, H, I and empty squares at D, E, F, for a total of 2 ways after counting symmetries.

Applying PIE, we get that the total number of boards with at least one pair of knights attacking each other is

$$8 \cdot 2^5 - 4 \cdot (2^3 + 2^3 + 2^2 + 2^3) + 4 \cdot 2 \cdot 2 - 2 = 158.$$

Then the probability the 3×3 board has a pair of knights attacking each other is $\frac{158}{2^9} = \boxed{\frac{79}{256}}$.

28. [**19**] Compute the number of functions $f : \{1, 2, \ldots, 9\} \to \{1, 2, \ldots, 9\}$ which satisfy $f(f(f(f(f(x))))) = x$ for each $x \in \{1, 2, \ldots, 9\}$.

Proposed by: Evan Chen

Answer: $\boxed{3025}$

All cycles lengths in the permutation must divide 5, which is a prime number. Either $f(x) = x$ for all x, or there exists exactly one permutation cycle of length 5. In the latter case, there are $\binom{9}{5}$ ways to choose which numbers are in the cycle and 4! ways to create the cycle. The answer is thus $1 + \binom{9}{5} \cdot 4! = 3025$.

29. [**19**] Consider a sequence x_n such that $x_1 = x_2 = 1$, $x_3 = \frac{2}{3}$. Suppose that $x_n = \dfrac{x_{n-1}^2 x_{n-2}}{2x_{n-2}^2 - x_{n-1}x_{n-3}}$ for all $n \geq 4$. Find the least n such that $x_n \leq \frac{1}{10^6}$.

Proposed by: Mehtaab Sawhney

Answer: $\boxed{13}$

The recursion simplifies to $\frac{x_{n-1}}{x_n} + \frac{x_{n-3}}{x_{n-2}} = 2\frac{x_{n-2}}{x_{n-1}}$. So if we set $y_n = \frac{x_{n-1}}{x_n}$ for $n \geq 2$ then we have $y_n - y_{n-1} = y_{n-1} - y_{n-2}$ for $n \geq 3$, which means that $\{y_n\}$ is an arithmetic sequence. From the starting values we have $y_2 = 1, y_3 = \frac{3}{2}$, so $y_n = \frac{n}{2}$ for all n. (This means that $x_n = \frac{2^{n-1}}{n!}$.) Since $\frac{x_1}{x_n} = y_2 y_3 \cdots y_n$, it suffices to find the minimal n such that the RHS is at least 10^6. Note that

$$y_2 y_3 \cdots y_{12} = 1 \cdot (1.5 \cdot 2 \cdot 2.5 \cdot 3 \cdot 3.5) \cdot (4 \cdot 4.5 \cdot 5 \cdot 5.5 \cdot 6) < 2.5^5 \cdot 5^5 = 12.5^5 < 200^2 \cdot 12.5 = 500000 < 10^6,$$

while

$$y_2 y_3 \cdots y_{13} = 1 \cdot (1.5 \cdot 2 \cdot 2.5 \cdot 3) \cdot (3.5 \cdot 4 \cdot 4.5) \cdot (5 \cdot 5.5 \cdot 6 \cdot 6.5) > 20 \cdot 60 \cdot 900 = 1080000 > 10^6,$$

so the answer is 13.

30. [**19**] Given complex number z, define sequence $z_0, z_1, z_2, \ldots$ as $z_0 = z$ and $z_{n+1} = 2z_n^2 + 2z_n$ for $n \geq 0$. Given that $z_{10} = 2017$, find the minimum possible value of $|z|$.

Proposed by: Yuan Yao

Answer: $\boxed{\frac{\sqrt[1024]{4035}-1}{2}}$

Define $w_n = z_n + \frac{1}{2}$, so $z_n = w_n - \frac{1}{2}$, and the original equation becomes

$$w_{n+1} - \frac{1}{2} = 2(w_n - \frac{1}{2})^2 + 2(w_n - \frac{1}{2}) = 2w_n^2 - \frac{1}{2},$$

which reduces to $w_{n+1} = 2w_n^2$. it is not difficult to show that

$$z_{10} + \frac{1}{2} = 2017 + \frac{1}{2} = \frac{4035}{2} = w_{10} = 2^{1023} w_0^{1024},$$

and thus $w_0 = \frac{\sqrt[1024]{4035}}{2}\omega_{1024}$, where ω_{1024} is one of the 1024^{th} roots of unity. Since $|w_0| = \frac{\sqrt[1024]{4035}}{2} > \frac{1}{2}$, to minimize the magnitude of $z = w_0 - \frac{1}{2}$, we need $\omega_{1024} = -1$, which gives $|z| = \frac{\sqrt[1024]{4035}-1}{2}$.

31. [**24**] In unit square $ABCD$, points E, F, G are chosen on side BC, CD, DA respectively such that AE is perpendicular to EF and EF is perpendicular to FG. Given that $GA = \frac{404}{1331}$, find all possible values of the length of BE.

Proposed by: Yuan Yao

Answer: $\boxed{\frac{9}{11}}$

Let $BE = x$, then since triangles ABE, ECF, FDG are all similar, we have $CE = 1 - x, CF = x(1-x), FD = 1 - x(1-x), DG = x - x^2(1-x), GA = 1 - x + x^2(1-x) = (1-x)(x^2+1)$, therefore it remains to solve the equation

$$(1-x)(x^2+1) = \frac{404}{1331}.$$

We first seek rational solutions $x = \frac{p}{q}$ for relatively prime positive integers p, q. Therefore we have $\frac{(q-p)(p^2+q^2)}{q^3} = \frac{404}{1331}$. Since both $q-p$ and p^2+q^2 are relatively prime to q^3, we have $q^3 = 1331 \Rightarrow q = 11$, so $(11-p)(p^2+121) = 404 = 2^2 \cdot 101$, and it is not difficult to see that $p = 9$ is the only integral solution. We can therefore rewrite the original equation as

$$(x - \frac{9}{11})(x^2 - \frac{2}{11}x + \frac{103}{121}) = 0.$$

It is not difficult to check that the quadratic factor has no zeroes, therefore $BE = x = \frac{9}{11}$ is the only solution.

32. [**24**] Let P be a polynomial with integer coefficients such that $P(0) + P(90) = 2018$. Find the least possible value for $|P(20) + P(70)|$.

Proposed by: Michael Tang

Answer: $\boxed{782}$

First, note that $P(x) = x^2 - 3041$ satisfy the condition and gives $|P(70)+P(20)| = |4900+400-6082| = 782$. To show that 782 is the minimum, we show $2800 \mid P(90) - P(70) - P(20) + P(0)$ for every P, since -782 is the only number in the range $[-782, 782]$ that is congruent to 2018 modulo 2800.

Proof: It suffices to show that $2800 \mid 90^n - 70^n - 20^n + 0^n$ for every $n \geq 0$ (having $0^0 = 1$).

Let $Q(n) = 90^n - 70^n - 20^n + 0^n$, then we note that $Q(0) = Q(1) = 0, Q(2) = 2800$, and $Q(3) = (9^3 - 7^3 - 2^3) \cdot 10^3 = 378000 = 135 \cdot 2800$. For $n \geq 4$, we note that 400 divides 10^4, and $90^n + 0^n \equiv 70^n + 20^n \pmod 7$. Therefore $2800 \mid Q(n)$ for all n.

33. [**24**] Tetrahedron $ABCD$ with volume 1 is inscribed in circumsphere ω such that $AB = AC = AD = 2$ and $BC \cdot CD \cdot DB = 16$. Find the radius of ω.

Proposed by: Caleb He

Answer: $\boxed{\frac{5}{3}}$

Let X be the foot of the perpendicular from A to ΔBCD. Since $AB = AC = AD$, it follows that X is the circumcenter of ΔBCD. Denote $XB = XC = XD = r$. By the Pythagorean Theorem on ΔABX, we have $AX = \sqrt{4 - r^2}$. Now, from the extended law of sines on ΔBCD, we have the well-known identity

$$\frac{BC \cdot CD \cdot DB}{4r} = [BCD],$$

where $[BCD]$ denotes the area of ΔBCD. However, we have

$$V = \frac{1}{3} AX \cdot [BCD],$$

where V is the volume of $ABCD$, which yields the expression

$$[BCD] = \frac{3}{\sqrt{4 - r^2}}.$$

Now, given that $BC \cdot CD \cdot DB = 16$, we have

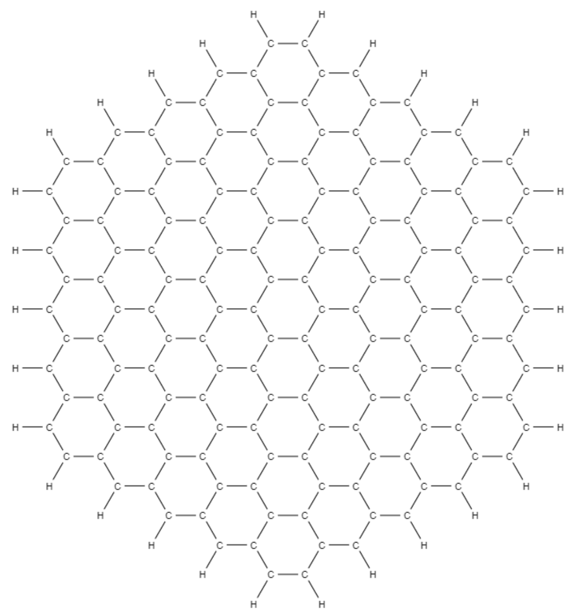
$$\frac{4}{r} = \frac{3}{\sqrt{4 - r^2}}.$$

Solving, we get $r = \frac{8}{5}$. Now, let O be the center of ω. Since $OB = OC = OD$, it follows that the foot of the perpendicular from O to ΔBCD must also be the circumcenter of ΔBCD, which is X. Thus, A, X, O are collinear. Let R be the radius of ω. Then we have

$$\begin{aligned} R &= OA \\ &= OX + XA \\ &= \sqrt{R^2 - r^2} + \sqrt{4 - r^2} \\ &= \sqrt{R^2 - \frac{64}{25}} + \frac{6}{5}. \end{aligned}$$

Solving, we get $R = \frac{5}{3}$. (Note: solving for R from $OA = OX - XA$ gives a negative value for R.)

34. [**30**] The skeletal structure of circumcircumcircumcoronene, a hydrocarbon with the chemical formula $C_{150}H_{30}$, is shown below.



Each line segment between two atoms is at least a single bond. However, since each carbon (C) requires exactly four bonds connected to it and each hydrogen (H) requires exactly one bond, some of the line segments are actually double bonds. How many arrangements of single/double bonds are there such that the above requirements are satisfied? If the correct answer is C and your answer is A, you get $\max\left(\left\lfloor 30\left(1-\left|\log_{\log_2 C}\frac{A}{C}\right|\right)\right\rfloor,0\right)$ points.

Proposed by: Yuan Yao

Answer: 267227532

The problem is equivalent to the one in OEIS A008793, a.k.a. "number of ways to tile hexagon of edge n with diamonds of side 1." Notice that there is a bjiection between such a tiling and the number of ways to stack some unit cubes alongside a corner of an $n \times n \times n$ box (see the Art of Problem Solving logo as an example, also known as 3-dimensional Young diagrams), where this problem $n = 5$. It is known that there are $\binom{2n}{n} = 252$ ways to stack one layer (since each way correspond a way to walk from a corner of a 5 by 5 grid to the opposite one), so $\frac{252^5}{5!} \approx 8 \times 10^9$ gives a somewhat loose upper bound (generate five layers and sort them by size, and hope that it will be a valid stack in general). This result can be improved by dividing out a reasonable constant factor after considering the probability that sorting by size indeed gives a valid stack (for example, it would be fair to guess that there is about a $\frac{1}{4!}$ chance that the first row of each layer will be in the right order, given that each row has a small role in determining the final size of the layer; dividing 24 from the previous result gives a very close guess). In general, a guess anywhere between 10^8 and 10^9 is a fair guess.

35. [**30**] Rebecca has twenty-four resistors, each with resistance 1 ohm. Every minute, she chooses any two resistors with resistance of a and b ohms respectively, and combine them into one by one of the following methods:

- Connect them in series, which produces a resistor with resistance of $a + b$ ohms;
- Connect them in parallel, which produces a resistor with resistance of $\frac{ab}{a+b}$ ohms;
- Short-circuit one of the two resistors, which produces a resistor with resistance of either a or b ohms.

Suppose that after twenty-three minutes, Rebecca has a single resistor with resistance R ohms. How many possible values are there for R? If the correct answer is C and your answer is A, you get $\max\left(\left\lfloor 30\left(1-\left|\log_{\log_2 C}\frac{A}{C}\right|\right)\right\rfloor,0\right)$ points.

Proposed by: Yuan Yao

Answer: $\boxed{1015080877}$

This is the same problem as in OEIS A153588. It is helpful to see (or guess) that neither the numerator or the denominator of the final resistance exceed the $(n+1)$-th Fibonacci number, which in this case is $F_{25} = 75025$, using concepts on the line of continued fractions. So $75025^2 \approx 5.6 \times 10^9$ is an upper bound for the total number, which is already close to the final answer. Multiplying by some constant factor to remove non-reduced fractions (such as $\frac{3}{4}$ to deal with parity) will improve this result.

36. [**30**] A box contains twelve balls, each of a different color. Every minute, Randall randomly draws a ball from the box, notes its color, and then *returns it to the box.* Consider the following two conditions:

(1) Some ball has been drawn at least twelve times (not necessarily consecutively).

(2) Every ball has been drawn at least once.

What is the probability that condition (1) is met *before* condition (2)? If the correct answer is C and your answer is A, you get $\max\left(\left\lfloor 30\left(1-\frac{1}{2}\left|\log_2 C - \log_2 A\right|\right)\right\rfloor, 0\right)$ points.

Proposed by: Yuan Yao

Answer: $\boxed{0.02236412255\ldots}$

Below is a python implementation to compute the probability, using the same method as the solution to the easier version (with three balls).

```
from fractions import Fraction

N = 12
probs = [{} for i in range((N-1)*(N-1)+2)]
prob1 = Fraction()
prob2 = Fraction()
init = tuple(0 for i in range(N))
probs[0][init] = Fraction(1,1)

for i in range((N-1)*(N-1)+1):
    for t in probs[i]:
        for j in range(N):
            val = probs[i][t] * Fraction(1,N)
            l = list(t)
            l[j] += 1
            l.sort()
            l = tuple(l)
            if (l[-1] == N):
                prob1 = prob1 + val
            elif (l[0] == 1):
                prob2 = prob2 + val
            else:
                probs[i+1][l] = probs[i+1].setdefault(l, Fraction()) + val

print(prob1)
```

Intuitively the probability should be quite small, since the distribution tends towards the second condition instead of the first. Indeed, the exact fraction is $p = \frac{M}{N}$, where

$$\begin{aligned} M =& 663659309086473387879121984765654681548533307869748367531 \\ & 919050571107782711246694886954585701687513519369602069583, \\ N =& 296751776202171713806564101986511242061620934987688694638 2 \\ & 1672067789922444492392280614561539198623553884143178743808. \end{aligned}$$

Note: This is a simplified variant of the Bingo Paradox, which is a phenomenon where horizontal bingos are significantly more frequent than vertical bingos. For more information, see `https://www.maa.org/sites/default/files/pdf/Mathhorizons/pdfs/The_Bingo_Paradox_MH_Sept17.pdf`.