HMMT November 2017

November 11, 2017

Team Round

1. [**15**] A positive integer k is called *powerful* if there are distinct positive integers p, q, r, s, t such that $p^2, q^3, r^5, s^7, t^{11}$ all divide k. Find the smallest powerful integer.

2. [**20**] How many sequences of integers $(a_1, \ldots, a_7)$ are there for which $-1 \leq a_i \leq 1$ for every i, and
$$a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 + a_5a_6 + a_6a_7 = 4?$$

3. [**25**] Michael writes down all the integers between 1 and N inclusive on a piece of paper and discovers that exactly 40% of them have leftmost digit 1. Given that $N > 2017$, find the smallest possible value of N.

4. [**30**] An equiangular hexagon has side lengths $1, 1, a, 1, 1, a$ in that order. Given that there exists a circle that intersects the hexagon at 12 distinct points, we have $M < a < N$ for some real numbers M and N. Determine the minimum possible value of the ratio $\frac{N}{M}$.

5. [**35**] Ashwin the frog is traveling on the xy-plane in a series of $2^{2017} - 1$ steps, starting at the origin. At the n^{th} step, if n is odd, then Ashwin jumps one unit to the right. If n is even, then Ashwin jumps m units up, where m is the greatest integer such that 2^m divides n. If Ashwin begins at the origin, what is the area of the polygon bounded by Ashwin's path, the line $x = 2^{2016}$, and the x-axis?

6. [**40**] Consider five-dimensional Cartesian space
$$\mathbb{R}^5 = \{(x_1, x_2, x_3, x_4, x_5) \mid x_i \in \mathbb{R}\},$$
and consider the hyperplanes with the following equations:
   - $x_i = x_j$ for every $1 \leq i < j \leq 5$;
   - $x_1 + x_2 + x_3 + x_4 + x_5 = -1$;
   - $x_1 + x_2 + x_3 + x_4 + x_5 = 0$;
   - $x_1 + x_2 + x_3 + x_4 + x_5 = 1$.

   Into how many regions do these hyperplanes divide $\mathbb{R}^5$?

7. [**50**] There are 12 students in a classroom; 6 of them are Democrats and 6 of them are Republicans. Every hour the students are randomly separated into four groups of three for political debates. If a group contains students from both parties, the minority in the group will change his/her political alignment to that of the majority at the end of the debate. What is the expected amount of time needed for all 12 students to have the same political alignment, in hours?

8. [**55**] Find the number of quadruples (a, b, c, d) of integers with absolute value at most 5 such that
$$(a^2 + b^2 + c^2 + d^2)^2 = (a + b + c + d)(a - b + c - d)((a - c)^2 + (b - d)^2).$$

9. [**60**] Let A, B, C, D be points chosen on a circle, in that order. Line BD is reflected over lines AB and DA to obtain lines ℓ_1 and ℓ_2 respectively. If lines ℓ_1, ℓ_2, and AC meet at a common point and if $AB = 4, BC = 3, CD = 2$, compute the length DA.

10. [**70**] Yannick has a bicycle lock with a 4-digit passcode whose digits are between 0 and 9 inclusive. (Leading zeroes are allowed.) The dials on the lock is currently set at 0000. To unlock the lock, every second he picks a contiguous set of dials, and increases or decreases all of them by one, until the dials are set to the passcode. For example, after the first second the dials could be set to $1100, 0010$, or 9999, but not 0909 or 0190. (The digits on each dial are cyclic, so increasing 9 gives 0, and decreasing 0 gives 9.) Let the *complexity* of a passcode be the minimum number of seconds he needs to unlock the lock. What is the maximum possible complexity of a passcode, and how many passcodes have this maximum complexity? Express the two answers as an ordered pair.