HMMT November 2017

November 11, 2017

Team Round

1. [**15**] A positive integer k is called *powerful* if there are distinct positive integers p, q, r, s, t such that $p^2, q^3, r^5, s^7, t^{11}$ all divide k. Find the smallest powerful integer.

   Proposed by: Yuan Yao

   Answer: $\boxed{1024}$

   First of all, 1024 is powerful because it can be divided evenly by $16^2 = 256, 8^3 = 512, 4^5 = 1024, 2^7 = 128$, and $1^{11} = 1$.

   Now we show that 1024 is the smallest powerful number. Since $s \neq t$, at least one of them is at least 2. If $t \geq 2$ or $s \geq 3$, then we need the number to be divisible by at least $2^{11} = 2048$ or $3^7 = 2187$, which both exceed 1024, so we must have $s = 2$ and $t = 1$. If $r = 3$, then the number must be divisible by $3^5 = 243$ and $2^7 = 128$, which means that the number is at least $243 \cdot 128 > 1024$, so $r \geq 4$, and the number is at least $4^5 = 1024$. Therefore the smallest powerful number is indeed 1024.

2. [**20**] How many sequences of integers $(a_1, \ldots, a_7)$ are there for which $-1 \leq a_i \leq 1$ for every i, and

$$a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 + a_5a_6 + a_6a_7 = 4?$$

   Proposed by: mendel keller

   Answer: $\boxed{38}$

   For $i = 1, 2, \ldots, 6$, let $b_i = a_ia_{i+1}$. From the problem condition each of $b_1, b_2, \ldots, b_6$ can only be $-1, 0$, or 1. Since the sum of these six numbers is 4, either there are five 1s and a -1 or there are four 1s and two 0s.

   In the first case, there are 6 ways to choose i such that $b_i = -1$. Once that is fixed, determining the value of a_1 (one of 1 and -1) will determine the value of all the remaining a_i's, so there are $6 \cdot 2 = 12$ possible ways in this case.

   In the second case, since if one of b_2, b_3, b_4, b_5 is zero, then one of the adjacent term to this zero term must also be zero. Therefore the two zeroes must be next to each other or be b_1 and b_6.

   If $b_1 = b_2 = 0$, then a_2 must be zero. a_1's value doesn't matter, and $a_3, a_4, \ldots, a_7$ must have the same sign. The same goes for $b_5 = b_6 = 0$, giving $3 \cdot 2 \cdot 2 = 12$ possibilities in these two cases.

   If $b_i = b_{i+1} = 0$ for $i = 2, 3, 4$, then a_{i+1} must be zero. Moreover, $a_1, a_2, \ldots, a_i$ must have the same sign, and so do $a_{i+2}, \ldots, a_7$. this gives $2 \cdot 2 \cdot 3 = 12$ possibilities in these three cases.

   If $b_1 = b_6 = 0$, then $a_1 = a_7 = 0$. Also, $a_2, a_3, \ldots, a_6$ must have the same sign so there are 2 possibilities.

   Combining these cases gives $12 + 12 + 12 + 2 = 38$ possible sequences in total.

3. [**25**] Michael writes down all the integers between 1 and N inclusive on a piece of paper and discovers that exactly 40% of them have leftmost digit 1. Given that $N > 2017$, find the smallest possible value of N.

   Proposed by: Michael Tang

   Answer: $\boxed{1{,}481{,}480}$

   Let d be the number of digits of N. Suppose that N does not itself have leftmost digit 1. Then the number of integers $1, 2, \ldots, N$ which have leftmost digit 1 is

$$1 + 10 + 10^2 + \ldots + 10^{d-1} = \frac{10^d - 1}{9},$$

   so we must have $\frac{10^d-1}{9} = \frac{2N}{5}$, or $5(10^d - 1) = 18N$. But the left-hand side is odd, so this is impossible.

Thus N must have leftmost digit 1. In this case, the number of integers $1, 2, \ldots, N$ which have leftmost digit 1 is

$$\begin{aligned} & 1 + 10 + 10^2 + \ldots + 10^{d-2} + (N - 10^{d-1} + 1) \\ =& \frac{10^{d-1} - 1}{9} + N - 10^{d-1} + 1 \\ =& N - 8\left(\frac{10^{d-1} - 1}{9}\right). \end{aligned}$$

Therefore we need $N - 8\left(\frac{10^{d-1}-1}{9}\right) = \frac{2}{5}N$, or $N = \frac{40}{3}\left(\frac{10^{d-1}-1}{9}\right)$. Then, $\frac{10^{d-1}-1}{9}$ must be divisible by 3. The base-ten representation of $\frac{10^{d-1}-1}{9}$ has $d-1$ ones, so $d-1$ must be divisible by 3. Both $d = 1$ and $d = 4$ make N less than 2017, but $d = 7$ gives the answer $N = \frac{40}{3}(111111) = \boxed{1481480}$.

4. [**30**] An equiangular hexagon has side lengths $1, 1, a, 1, 1, a$ in that order. Given that there exists a circle that intersects the hexagon at 12 distinct points, we have $M < a < N$ for some real numbers M and N. Determine the minimum possible value of the ratio $\frac{N}{M}$.

Proposed by: Yuan Yao

Answer: $\boxed{\frac{3}{\sqrt{3}-1} \text{ OR } \frac{3\sqrt{3}+3}{2}}$

We claim that the greatest possible value of M is $\sqrt{3} - 1$, whereas the least possible value of N is 3.

To begin, note that the condition requires the circle to intersect each side of the hexagon at two points *on its interior*. This implies that the center must be inside the hexagon as its projection onto all six sides must be on their interior. Suppose that the hexagon is $ABCDEF$, with $AB = BC = DE = EF = 1$, $CD = FA = a$, and the center O.

When $a \leq \sqrt{3} - 1$, we note that the distance from O to CD (which is $\frac{\sqrt{3}}{2}$) is greater than or equal to the distance from O to B or E (which is $\frac{a+1}{2}$). However, for the circle to intersect all six sides at two points each, the distance from the center of the circle to CD and to FA must be strictly less than that from the center to B and to E, because otherwise any circle that intersects CD and FA at two points each must include B or E on its boundary or interior, which will not satisfy the condition. WLOG assume that the center of the circle is closer to FA than to CD, including equality (in other words, the center is on the same side of BE as FA, possibly on BE itself), then note that the parabola with foci B and E and common directrix CD intersects on point O, which means that there does not exist a point in the hexagon on the same side of BE as FA that lies on the same side of both parabola as CD. This means that the center of the circle cannot be chosen.

When $a = \sqrt{3} - 1 + \epsilon$ for some very small real number $\epsilon > 0$, the circle with center O and radius $r = \frac{\sqrt{3}}{2}$ intersects sides AB, BC, DE, EF at two points each and is tangent to CD and FA on their interior. Therefore, there exists a real number $\epsilon' > 0$ such that the circle with center O and radius $r' = r + \epsilon'$ satisfy the requirement.

When $a \geq 3$, we note that the projection of BF onto BC has length $|\frac{1}{2} - \frac{a}{2}| \geq 1$, which means that the projection of F onto side BC is not on its interior, and the same goes for side EF onto BC. However, for a circle to intersect both BC and EF at two points, the projection of center of the circle onto the two sides must be on their interior, which cannot happen in this case.

When $a = 3 - \epsilon$ for some very small real number $\epsilon > 0$, a circle with center O and radius $r = \frac{\sqrt{3}}{4}(a+1)$ intersects AF and CD at two points each and is tangent to all four other sides on their interior. Therefore, there exists a real number $\epsilon' > 0$ such that the circle with center O and radius $r' = r + \epsilon'$ satisfy the requirement.

With $M \leq \sqrt{3} - 1$ and $N \geq 3$, we have $\frac{N}{M} \geq \frac{3}{\sqrt{3}-1} = \frac{3\sqrt{3}+3}{2}$, which is our answer.

5. [**35**] Ashwin the frog is traveling on the xy-plane in a series of $2^{2017} - 1$ steps, starting at the origin. At the n^{th} step, if n is odd, then Ashwin jumps one unit to the right. If n is even, then Ashwin jumps m units up, where m is the greatest integer such that 2^m divides n. If Ashwin begins at the origin, what is the area of the polygon bounded by Ashwin's path, the line $x = 2^{2016}$, and the x-axis?

Proposed by: Nikhil Reddy

Answer: $\boxed{2^{2015} \cdot (2^{2017} - 2018)}$

Notice that since $v_2(x) = v_2(2^{2017} - x)$, the path divides the rectangle bonded by the coordinate axes and the two lines passing through Ashwin's final location parallel to the axes. The answer is therefore half of the product of the coordinates of Ashwin's final coordinates. The x-coordinate is the number of odd number steps (which is 2^{2016}). The y-coordinate is the number of total powers of 2 in $(2^{2017} - 1)!$. The final answer is therefore $2^{2015} \cdot (2^{2017} - 2018)$.

6. [**40**] Consider five-dimensional Cartesian space

$$\mathbb{R}^5 = \{(x_1, x_2, x_3, x_4, x_5) \mid x_i \in \mathbb{R}\},$$

and consider the hyperplanes with the following equations:

- $x_i = x_j$ for every $1 \le i < j \le 5$;
- $x_1 + x_2 + x_3 + x_4 + x_5 = -1$;
- $x_1 + x_2 + x_3 + x_4 + x_5 = 0$;
- $x_1 + x_2 + x_3 + x_4 + x_5 = 1$.

Into how many regions do these hyperplanes divide $\mathbb{R}^5$?

Proposed by: Mehtaab Sawhney

Answer: $\boxed{480}$

(Joint with Junyao Peng)

Note that given a set of plane equations $P_i(x_1, x_2, x_3, x_4, x_5) = 0$, for $i = 1, 2, \ldots, n$, each region that the planes separate the space into correspond to a n-tuple of -1 and 1, representing the sign of $P_1, P_2, \ldots P_n$ for all points in that region.

Therefore, the first set of planes separate the space into $5! = 120$ regions, with each region representing an ordering of the five coordinates by numerical size. Moreover, the next three planes are parallel to each other and perpendicular to all planes in the first set, so these three planes separate each region into 4. Therefore, a total of $4 \cdot 120 = 480$ regions is created.

7. [**50**] There are 12 students in a classroom; 6 of them are Democrats and 6 of them are Republicans. Every hour the students are randomly separated into four groups of three for political debates. If a group contains students from both parties, the minority in the group will change his/her political alignment to that of the majority at the end of the debate. What is the expected amount of time needed for all 12 students to have the same political alignment, in hours?

Proposed by: Yuan Yao

Answer: $\boxed{\frac{341}{54}}$

When the party distribution is $6 - 6$, the situation can change (to $3 - 9$) only when a group of three contains three people from the same party, and the remaining three are distributed evenly across the other three groups (to be converted).

To compute the probability, we assume that the groups and the members of the group are ordered (so there are $12!$ ways of grouping). There are 2 ways to choose the party, 4 ways to choose the group, $6 \cdot 5 \cdot 4$ ways to choose the three members of the group, $9 \cdot 6 \cdot 3$ ways to place the other three members of the party, and $6!$ ways to fill in the members of the other party. The probability is then

$$\frac{2 \cdot 4 \cdot 6 \cdot 5 \cdot 4 \cdot 9 \cdot 6 \cdot 3 \cdot 6!}{12!} = \frac{2 \cdot 4 \cdot 6 \cdot 5 \cdot 4 \cdot 9 \cdot 6 \cdot 3}{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7} = \frac{18}{77}.$$

This means that the shift in distribution will happen in $\frac{77}{18}$ hours on average.

When the distribution is $3-9$, the situation can change (to $0-12$) only when the three members of the minority party are all in different groups. Using the similar method as above, there are $12 \cdot 9 \cdot 6$ ways to place the three members and 9! ways to place the rest, so the probability is

$$\frac{12 \cdot 9 \cdot 6 \cdot 9!}{12!} = \frac{12 \cdot 9 \cdot 6}{12 \cdot 11 \cdot 10} = \frac{27}{55}.$$

This means that the shift in distribution will happen in $\frac{55}{27}$ hours on average.

By linearity of expectation, we can add up the two results and get that the expected value is $\frac{77}{18} + \frac{55}{27} = \frac{341}{55}$ hours.

8. [**55**] Find the number of quadruples (a, b, c, d) of integers with absolute value at most 5 such that

$$(a^2+b^2+c^2+d^2)^2 = (a+b+c+d)(a-b+c-d)((a-c)^2+(b-d)^2).$$

Proposed by: Mehtaab Sawhney

Answer: $\boxed{49}$

Let $x = a+c$, $y = a-c$, $w = b+d$, and $z = b-d$. Then

$$(w^2+x^2+y^2+z^2)^2 = 4(x^2-w^2)(y^2+z^2)$$

and since $|x^2+w^2| \geq |x^2-w^2|$ it follows that $w=0$ or $y=z=0$. Now $y=z=0$ implies $a=b=c=d=0$. Now $w=0$ gives $b=-d$. Then for equality to hold $x^2 = y^2+z^2$. This is equivalent to $ac = b^2$, which includes the previous case. It suffices to count the number of triples (a, b, c) that satisfy the equation.

When $b=0$, either a or c is zero, which gives $11+11-1=21$ triples.

When $b=\pm 1$, we have $|a|=|c|=1$ and a, c have the same sign, for $2 \cdot 2 = 4$ triples.

When $b = \pm 2$, we have $(a, c) = (1, 4), (2, 2), (4, 1)$ or their negatives, for $2 \cdot 6 = 12$ triples.

When $b = \pm 3, \pm 4, \pm 5$, we have $|a| = |b| = |c|$ and a, c have the same sign, for $6 \cdot 2 = 12$ triples.

So in total there are $21+4+12+12 = 49$ solutions.

9. [**60**] Let A, B, C, D be points chosen on a circle, in that order. Line BD is reflected over lines AB and DA to obtain lines ℓ_1 and ℓ_2 respectively. If lines ℓ_1, ℓ_2, and AC meet at a common point and if $AB = 4, BC = 3, CD = 2$, compute the length DA.

Proposed by: Ashwin Sah

Answer: $\boxed{\sqrt{21}}$

Let the common point be E. Then since lines BE and BD are symmetric about line BA, BA is an exterior bisector of $\angle DBE$, and similarly DA is also an exterior bisector of $\angle BDE$. Therefore A is the E-excenter of triangle BDE and thus lie on the interior bisector of $\angle BED$. Since C lies on line AE, and by the fact that A, B, C, D are concyclic, we get that $\angle ABC + \angle ADC = 180°$, which implies $\angle DBC + \angle BDC = \frac{1}{2}(\angle DBE + \angle BDE)$, so C is the incenter of triangle BDE. Thus $\angle ABC = \angle CDA = \frac{\pi}{2}$, and thus $DA^2 = AC^2 - CD^2 = AB^2 + BC^2 - CD^2 = 3^2 + 4^2 - 2^2 = 21$. The length of DA is then $\sqrt{21}$.

10. [**70**] Yannick has a bicycle lock with a 4-digit passcode whose digits are between 0 and 9 inclusive. (Leading zeroes are allowed.) The dials on the lock is currently set at 0000. To unlock the lock, every second he picks a contiguous set of dials, and increases or decreases all of them by one, until the dials are set to the passcode. For example, after the first second the dials could be set to 1100, 0010, or 9999, but not 0909 or 0190. (The digits on each dial are cyclic, so increasing 9 gives 0, and decreasing 0 gives 9.) Let the *complexity* of a passcode be the minimum number of seconds he needs to unlock the lock. What is the maximum possible complexity of a passcode, and how many passcodes have this maximum complexity? Express the two answers as an ordered pair.

Proposed by: Yuan Yao

Answer: $\boxed{(12,2)}$

To simplify the solution, we instead consider the equivalent problem of reducing a passcode to 0000 using the given move.

Given a passcode $a_1a_2a_3a_4$, define a *differential* of the passcode to be a quintuple $(d_1, d_2, d_3, d_4, d_5)$ such that $d_i \equiv a_i - a_{i-1} \pmod{10}$ for $i = 1, 2, 3, 4, 5$, where we define $a_0 = a_5 = 0$.

Claim 1: For any passcode, there exists a differential that satisfies the following two conditions:

- $d_1 + d_2 + d_3 + d_4 + d_5 = 0$;
- The range (difference between the maximum and minimum) of these five numbers is at most 10.

Proof: We first see that the differential defined by $d_i = a_i - a_{i-1}$ satisfy the first condition since the sum of the five numbers is $a_5 - a_0 = 0$. Suppose that a differential satisfying the first condition has a range greater than 10, then we take one of the largest number d_m and one of the smallest number d_n (where $d_m - d_n > 10$), and replace the former by $d_m - 10$ and the latter by $d_n + 10$. This will either reduce the range or reduce the number of maximal and minimal numbers, so the process will terminate after finitely many iterations. Thus we can find a differential that satisfies both conditions.

(Note: we call such a differential a *standard differential* from now on, although it is important to remember that there may be more than one standard differential for one passcode. As a corollary, all of the numbers in a standard differential must be in the range $[-9, 9]$, as a number greater than 9 will be more than 10 away from a negative number, and similar for a number smaller than -9.)

Given a passcode, we define the *magnitude* of one of its standard differentials to be the sum of all the positive values in the differential (which is also the absolute value of the sum all the negative values).

Claim 2: The magnitude of the a passcode's standard differential is equal to the passcode's complexity.

Proof: Obviously 0000 is the only passcode with complexity 0 whose standard differential has magnitude 0. Suppose that the magnitude of a standard differential is M, then it suffices show that the magnitude can be reduced to 0 in M moves, and that the magnitude can only decrease by at most 1 with each move.

The first part can be shown via the following algorithm. When the magnitude is not zero, there must be a positive number d_i and a negative number d_j. WLOG assume that $i < j$, then after taking the dials $a_i, a_{i+1}, \ldots, a_{j-1}$ and decreases them all by 1, d_i decreases by 1 and d_j increases by 1. This will decrease the magnitude by 1 (and the differential remains standard), so by repeating this process M times we can bring the magnitude down to 0.

For the second part, we assume WLOG that we take the dials $a_i, a_{i+1}, \ldots, a_{j-1}$ and decrease them all by 1, and then d_i is replaced by $d_i' = d_i - 1$ and $d_j' = d_j + 1$. If the differential remains standard, then the magnitude decreases by 1 (when $d_i > 0$ and $d_j < 0$), remains the same (when either $d_i \leq 0$ and $d_j < 0$ or $d_i > 0$ and $d_j \geq 0$), or increases by 1 (when $d_i \leq 0$ and $d_j \geq 0$).

In the latter two cases, it is possible that the differential is no longer standard.

If the magnitude previously remained the same (WLOG suppose that $d_i > 0$ and $d_j \geq 0$), then there exists a negative d_k that is minimal such that $d_j - d_k = 10$ and now d_j' is the unique maximum. Replacing d_k by $d_k + 10 = d_j$ and d_j' by $d_j' - 10 = d_k + 1$ will reduce the magnitude by 1, and the new differential will be standard because the unique maximum d_j' is no longer present and the minimum is now either d_k or $d_k + 1$. This means that the magnitude decreases by at most 1.

If the magnitude previously increased by 1 (when $d_i \leq 0$ and $d_j \geq 0$), then there exists either a negative d_k that is (previously) minimal such that $d_j - d_k = 10$, or a positive d_l that is (previously) maximal such that $d_l - d_i = 10$, or both. By similar logic as the previous case, replacing d_k by $d_k + 10$ and d_j' by $d_j' - 10$, or replacing d_l by $d_l - 10$ and d_i' by $d_i' + 10$ (or both, if both d_k and d_l exist). will decrease the magnitude by 1, and ensure that the new differential is standard. The replacement will decrease the current magnitude by at most 2, this means that the original magnitude decreases by at most 1 in total.

These considerations finishes the second part and therefore the proof.

With this claim, we also see that the magnitudes of all possible standard differentials of a given passcode are the same, so the choice of the differential is irrelevant.

We can now proceed to find the maximum possible complexity. Suppose that there are m positive numbers and n negative numbers in the differential, and suppose that the maximum and the minimum are x and $-y$ respectively. Since the sum of all positive numbers is at most mx and the absolute value of the sum of all negative numbers is at most ny, the complexity is at most $C = \min(mx, ny)$. It suffices to maximize C under the condition that $m + n \leq 5$ and $x + y = x - (-y) \leq 10$. It is not difficult to see (via casework) that the maximal C is 12, achieved by $m = 2, n = 3, x = 6, y = 4$ or $m = 3, n = 2, x = 4, y = 6$. In the first case, the digits must increase from 0 by 6 twice and decreases by 4 three times (and reduced modulo 10), which gives the passcode 6284; in the second case the digits increase by 4 three times and decreases by 6 twice instead, which gives the passcode 4826. Since all inequalities are tight, these two passcodes are the only ones that has the maximal complexity of 12.