HMMT November 2017

November 11, 2017

Theme Round

Based on a true story.

1. **T**wo ordered pairs (a, b) and (c, d), where a, b, c, d are real numbers, form a basis of the coordinate plane if $ad \neq bc$. Determine the number of ordered quadruples (a, b, c, d) of integers between 1 and 3 inclusive for which (a, b) and (c, d) form a basis for the coordinate plane.

2. **H**orizontal parallel segments $AB = 10$ and $CD = 15$ are the bases of trapezoid $ABCD$. Circle γ of radius 6 has center within the trapezoid and is tangent to sides AB, BC, and DA. If side CD cuts out an arc of γ measuring $120°$, find the area of $ABCD$.

3. **E**milia wishes to create a basic solution with 7% hydroxide (OH) ions. She has three solutions of different bases available: 10% rubidium hydroxide (Rb(OH)), 8% cesium hydroxide (Cs(OH)), and 5% francium hydroxide (Fr(OH)). (The Rb(OH) solution has both 10% Rb ions and 10% OH ions, and similar for the other solutions.) Since francium is highly radioactive, its concentration in the final solution should not exceed 2%. What is the highest possible concentration of rubidium in her solution?

4. **M**ary has a sequence $m_2, m_3, m_4, \ldots$, such that for each $b \geq 2$, m_b is the least positive integer m for which none of the base-b logarithms $\log_b(m)$, $\log_b(m+1)$, $\ldots$, $\log_b(m+2017)$ are integers. Find the largest number in her sequence.

5. **E**ach of the integers $1, 2, \ldots, 729$ is written in its base-3 representation without leading zeroes. The numbers are then joined together in that order to form a continuous string of digits: $12101112202122\ldots$. How many times in this string does the substring 012 appear?

6. **R**thea, a distant planet, is home to creatures whose DNA consists of two (distinguishable) strands of bases with a fixed orientation. Each base is one of the letters H, M, N, T, and each strand consists of a sequence of five bases, thus forming five pairs. Due to the chemical properties of the bases, each pair must consist of distinct bases. Also, the bases H and M cannot appear next to each other on the same strand; the same is true for N and T. How many possible DNA sequences are there on Rthea?

7. **O**n a blackboard a stranger writes the values of $s_7(n)^2$ for $n = 0, 1, \ldots, 7^{20} - 1$, where $s_7(n)$ denotes the sum of digits of n in base 7. Compute the average value of all the numbers on the board.

8. **U**ndecillion years ago in a galaxy far, far away, there were four space stations in the three-dimensional space, each pair spaced 1 light year away from each other. Admiral Ackbar wanted to establish a base somewhere in space such that the sum of squares of the distances from the base to each of the stations does not exceed 15 square light years. (The sizes of the space stations and the base are negligible.) Determine the volume, in cubic light years, of the set of all possible locations for the Admiral's base.

9. **N**ew this year at HMNT: the exciting game of *RNG baseball*! In RNG baseball, a team of infinitely many people play on a square field, with a base at each vertex; in particular, one of the bases is called the *home base*. Every turn, a new player stands at home base and chooses a number n uniformly at random from $\{0, 1, 2, 3, 4\}$. Then, the following occurs:

   - If $n > 0$, then the player and everyone else currently on the field moves (counterclockwise) around the square by n bases. However, if in doing so a player returns to or moves past the home base, he/she leaves the field immediately and the team scores one point.
   - If $n = 0$ (a strikeout), then the game ends immediately; the team does not score any more points.

   What is the expected number of points that a given team will score in this game?

10. **D**enote $\phi = \frac{1+\sqrt{5}}{2}$ and consider the set of all finite binary strings without leading zeroes. Each string S has a "base-ϕ" value $p(S)$. For example, $p(1101) = \phi^3 + \phi^2 + 1$. For any positive integer n, let $f(n)$ be the number of such strings S that satisfy $p(S) = \frac{\phi^{48n}-1}{\phi^{48}-1}$. The sequence of fractions $\frac{f(n+1)}{f(n)}$ approaches a real number c as n goes to infinity. Determine the value of c.