HMMT November 2018

November 10, 2018

General Round

1. What is the largest factor of 130000 that does not contain the digit 0 or 5?

   Proposed by: Farrell Eldrian Wu

   Answer: $\boxed{26}$

   If the number is a multiple of 5, then its units digit will be either 0 or 5. Hence, the largest such number must have no factors of 5.

   We have $130000 = 2^4 \cdot 5^4 \cdot 13$. Removing every factor of 5, we get that our number must be a factor of $2^4 \cdot 13 = 208$.

   If our number contains a factor of 13, we cancel the factor of 2 from 208, 104, and 52 until we get 26. Otherwise, the largest number we can have is $2^4 = 16$. We conclude that the answer is 26.

2. Twenty-seven players are randomly split into three teams of nine. Given that Zack is on a different team from Mihir and Mihir is on a different team from Andrew, what is the probability that Zack and Andrew are on the same team?

   Proposed by: Yuan Yao

   Answer: $\boxed{\frac{8}{17}}$

   Once we have assigned Zack and Mihir teams, there are 8 spots for more players on Zack's team and 9 for more players on the third team. Andrew is equally likely to occupy any of these spots, so our answer is $\frac{8}{17}$.

3. A square in the xy-plane has area A, and three of its vertices have x-coordinates $2, 0,$ and 18 in some order. Find the sum of all possible values of A.

   Proposed by: Yuan Yao

   Answer: $\boxed{1168}$

   More generally, suppose three vertices of the square lie on lines $y = y_1, y = y_2, y = y_3$. One of these vertices must be adjacent to two others. If that vertex is on $y = y_1$ and the other two are on $y = y_2$ and $y = y_3$, then we can use the Pythagorean theorem to get that the square of the side length is $(y_2 - y_1)^2 + (y_3 - y_1)^2$.

   For $(y_1, y_2, y_3) = (2, 0, 18)$, the possibilities are $2^2 + 16^2$, $2^2 + 18^2$, $16^2 + 18^2$, so the sum is $2(2^2 + 16^2 + 18^2) = 2(4 + 256 + 324) = 1168$.

4. Find the number of eight-digit positive integers that are multiples of 9 and have all distinct digits.

   Proposed by: James Lin

   Answer: $\boxed{181440}$

   Note that $0 + 1 + \cdots + 9 = 45$. Consider the two unused digits, which must then add up to 9. If it's 0 and 9, there are $8 \cdot 7!$ ways to finish; otherwise, each of the other four pairs give $7 \cdot 7!$ ways to finish, since 0 cannot be the first digit. This gives a total of $36 \cdot 7! = 181440$.

5. Compute the smallest positive integer n for which

   $$\sqrt{100 + \sqrt{n}} + \sqrt{100 - \sqrt{n}}$$

   is an integer.

   Proposed by: Michael Tang

   Answer: $\boxed{6156}$

The number $\sqrt{100+\sqrt{n}}+\sqrt{100-\sqrt{n}}$ is a positive integer if and only if its square is a perfect square. We have

$$\begin{aligned}\left(\sqrt{100+\sqrt{n}}+\sqrt{100-\sqrt{n}}\right)^2 &= (100+\sqrt{n})+(100-\sqrt{n})+2\sqrt{(100+\sqrt{n})(100-\sqrt{n})}\\ &= 200+2\sqrt{10000-n}.\end{aligned}$$

To minimize n, we should maximize the value of this expression, given that it is a perfect square. For this expression to be a perfect square, $\sqrt{10000-n}$ must be an integer. Then $200+2\sqrt{10000-n}$ is even, and it is less than $200+2\sqrt{10000}=400=20^2$. Therefore, the greatest possible perfect square value of $200+2\sqrt{10000-n}$ is $18^2=324$. Solving

$$200+2\sqrt{10000-n}=324$$

for n gives the answer, $n=\boxed{6156}$.

6. Call a polygon *normal* if it can be inscribed in a unit circle. How many non-congruent normal polygons are there such that the square of each side length is a positive integer?

Proposed by: Yuan Yao

Answer: $\boxed{14}$

The side lengths of the polygon can only be from the set $\{1,\sqrt{2},\sqrt{3},2\}$, which take up $60°, 90°, 120°, 180°$ of the circle respectively. By working modulo 60 degrees we see that $\sqrt{2}$ must be used an even number of times. We now proceed to casework on the longest side of the polygon.

Case 1: If the longest side has length 2, then the remaining sides must contribute the remaining 180 degrees. There are 3 possibilities: $(1,1,1,2),(1,\sqrt{3},2),(\sqrt{2},\sqrt{2},2)$.

Case 2: If the longest side has length $\sqrt{3}$, then it takes up either $120°$ or $240°$ of the circle. In the former case we have 6 possibilities: $(1,1,1,1,\sqrt{3}),(1,\sqrt{2},\sqrt{2},\sqrt{3}),(\sqrt{2},1,\sqrt{2},\sqrt{3}),(1,1,\sqrt{3},\sqrt{3}),$ $(1,\sqrt{3},1,\sqrt{3}),(\sqrt{3},\sqrt{3},\sqrt{3})$. In the latter case there is only 1 possibility: $(1,1,\sqrt{3})$.

Case 3: If the longest side has length $\sqrt{2}$, then it shows up either twice or four times. In the former case we have 2 possibilities: $(1,1,1,\sqrt{2},\sqrt{2}),(1,1,\sqrt{2},1,\sqrt{2})$. In the latter case there is only 1 possibility: $(\sqrt{2},\sqrt{2},\sqrt{2},\sqrt{2})$.

Case 4: If all sides have length 1, then there is 1 possibility: $(1,1,1,1,1,1)$.

Adding up all cases, we have $3+6+1+2+1+1=14$ polygons.

7. Anders is solving a math problem, and he encounters the expression $\sqrt{15!}$. He attempts to simplify this radical by expressing it as $a\sqrt{b}$ where a and b are positive integers. The sum of all possible distinct values of ab can be expressed in the form $q\cdot 15!$ for some rational number q. Find q.

Proposed by: Nikhil Reddy

Answer: $\boxed{4}$

Note that $15!=2^{11}\cdot 3^6\cdot 5^3\cdot 7^2\cdot 11^1\cdot 13^1$. The possible a are thus precisely the factors of $2^5\cdot 3^3\cdot 5^1\cdot 7^1=30240$. Since $\frac{ab}{15!}=\frac{ab}{a^2b}=\frac{1}{a}$, we have

$$\begin{aligned}
q &= \frac{1}{15!} \sum_{\substack{a,b: \\ a\sqrt{b}=\sqrt{15!}}} ab \\
&= \sum_{a|30420} \frac{ab}{15!} \\
&= \sum_{a|30420} \frac{1}{a} \\
&= \left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}\right)\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right) \\
&= \left(\frac{63}{32}\right)\left(\frac{40}{27}\right)\left(\frac{6}{5}\right)\left(\frac{8}{7}\right) \\
&= 4.
\end{aligned}$$

8. Equilateral triangle ABC has circumcircle Ω. Points D and E are chosen on minor arcs AB and AC of Ω respectively such that $BC = DE$. Given that triangle ABE has area 3 and triangle ACD has area 4, find the area of triangle ABC.

   Proposed by: Yuan Yao

   Answer: $\boxed{\frac{37}{7}}$

   A rotation by $120°$ about the center of the circle will take ABE to BCD, so BCD has area 3. Let $AD = x, BD = y$, and observe that $\angle ADC = \angle CDB = 60°$. By Ptolemy's Theorem, $CD = x + y$. We have

$$4 = [ACD] = \frac{1}{2} AD \cdot CD \cdot \sin 60° = \frac{\sqrt{3}}{4} x(x+y)$$
$$3 = [BCD] = \frac{1}{2} BD \cdot CD \cdot \sin 60° = \frac{\sqrt{3}}{4} y(x+y)$$

   By dividing these equations find $x : y = 4 : 3$. Let $x = 4t, y = 3t$. Substitute this into the first equation to get $1 = \frac{\sqrt{3}}{4} \cdot 7t^2$. By the Law of Cosines,

$$AB^2 = x^2 + xy + y^2 = 37t^2.$$

   The area of ABC is then

$$\frac{AB^2\sqrt{3}}{4} = \frac{37}{7}.$$

9. 20 players are playing in a Super Smash Bros. Melee tournament. They are ranked $1 - 20$, and player n will always beat player m if $n < m$. Out of all possible tournaments where each player plays 18 distinct other players exactly once, one is chosen uniformly at random. Find the expected number of pairs of players that win the same number of games.

   Proposed by: Anders Olsen

   Answer: $\boxed{4}$

   Consider instead the complement of the tournament: The 10 possible matches that are not played. In order for each player to play 18 games in the tournament, each must appear once in these 10 unplayed matches. Players n and $n+1$ will win the same number of games if, in the matching, they are matched with each other, or n plays a player $a > n+1$ and $n+1$ plays a player $b < n$. (Note no other pairs of players can possibly win the same number of games.) The first happens with probability $\frac{1}{19}$ (as there are 19 players for player n to be paired with), and the second happens with probability $\frac{(n-1)(20-n-1)}{19 \cdot 17}$.

By linearity of expectation, the expected number of pairs of players winning the same number of games is the sum of these probabilities. We compute

$$\sum_{n=1}^{19}\left(\frac{1}{19}+\frac{(n-1)(20-n-1)}{323}\right)=\sum_{n=0}^{18}\left(\frac{1}{19}+\frac{n(18-n)}{323}\right)=1+\frac{\binom{19}{3}}{323}=4.$$

10. Real numbers x, y, and z are chosen from the interval $[-1,1]$ independently and uniformly at random. What is the probability that

$$|x|+|y|+|z|+|x+y+z|=|x+y|+|y+z|+|z+x|?$$

Proposed by: Yuan Yao

Answer: $\boxed{\frac{3}{8}}$

We assume that x, y, z are all nonzero, since the other case contributes zero to the total probabilty.

If x, y, z are all positive or all negative then the equation is obviously true. Otherwise, since flipping the signs of all three variables or permuting them does not change the equality, we assume WLOG that $x, y > 0$ and $z < 0$.

If $x+y+z > 0$, then the LHS of the original equation becomes $x+y-z+x+y=z=2x+2y$, and the RHS becomes $x+y+|x+z|+|y+z|$, so we need $|x+z|+|y+z|=x+y$. But this is impossible when $-x-y < z < 0$, since the equality is achieved only at the endpoints and all the values in between make the LHS smaller than the RHS. (This can be verified via simple casework.)

If $x+y+z < 0$, then $x+z, y+z < 0$ as well, so the LHS of the original equation becomes $x+y-z-x-y-z=-2z$ and the RHS becomes $x+y-x-z-y-z=-2z$. In this case, the equality holds true.

Thus, we seek the volume of all points $(x, y, z) \in [0,1]^3$ that satisfy $x+y-z<0$ (we flip the sign of z here for convenience). The equation $x+y-z=0$ represents a plane through the vertices $(1,0,1), (0,0,0), (0,1,1)$, and the desired region is the triangular pyramid, above the plane inside the unit cube, which has vertices $(1,0,1), (0,0,0), (0,1,1), (0,0,1)$. This pyramid has volume $\frac{1}{6}$.

So the total volume of all points in $[-1,1]^3$ that satisfy the equation is $2 \cdot 1+6 \cdot \frac{1}{6}=3$, out of $2^3=8$, so the probability is $\frac{3}{8}$.

Note: A more compact way to express the equality condition is that the equation holds true if and only if $xyz(x+y+z) \geq 0$.