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1. [**5**] A positive integer is called *primer* if it has a prime number of distinct prime factors. Find the smallest primer number.

2. [**5**] Pascal has a triangle. In the nth row, there are $n+1$ numbers $a_{n,0}, a_{n,1}, a_{n,2}, \ldots, a_{n,n}$ where $a_{n,0} = a_{n,n} = 1$. For all $1 \le k \le n-1$, $a_{n,k} = a_{n-1,k} - a_{n-1,k-1}$. What is the sum of all numbers in the 2018th row?

3. [**5**] An $n \times m$ maze is an $n \times m$ grid in which each cell is one of two things: a wall, or a blank. A maze is *solvable* if there exists a sequence of adjacent blank cells from the top left cell to the bottom right cell going through no walls. (In particular, the top left and bottom right cells must both be blank.) Determine the number of solvable 2×2 mazes.

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4. [**6**] Let a, b, c, n be positive real numbers such that $\frac{a+b}{a} = 3$, $\frac{b+c}{b} = 4$, and $\frac{c+a}{c} = n$. Find n.

5. [**6**] Jerry has ten distinguishable coins, each of which currently has heads facing up. He chooses one coin and flips it over, so it now has tails facing up. Then he picks another coin (possibly the same one as before) and flips it over. How many configurations of heads and tails are possible after these two flips?

6. [**6**] An equilateral hexagon with side length 1 has interior angles $90°, 120°, 150°, 90°, 120°, 150°$ in that order. Find its area.

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7. [**7**] At Easter-Egg Academy, each student has two eyes, each of which can be eggshell, cream, or cornsilk. It is known that 30% of the students have at least one eggshell eye, 40% of the students have at least one cream eye, and 50% of the students have at least one cornsilk eye. What percentage of the students at Easter-Egg Academy have two eyes of the same color?

8. [**7**] Pentagon $JAMES$ is such that $AM = SJ$ and the internal angles satisfy $\angle J = \angle A = \angle E = 90°$, and $\angle M = \angle S$. Given that there exists a diagonal of $JAMES$ that bisects its area, find the ratio of the shortest side of $JAMES$ to the longest side of $JAMES$.

9. [**7**] Farmer James has some strange animals. His hens have 2 heads and 8 legs, his peacocks have 3 heads and 9 legs, and his zombie hens have 6 heads and 12 legs. Farmer James counts 800 heads and 2018 legs on his farm. What is the number of animals that Farmer James has on his farm?

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10. [**8**] Abbot writes the letter A on the board. Every minute, he replaces every occurrence of A with AB and every occurrence of B with BA, hence creating a string that is twice as long. After 10 minutes, there are $2^{10} = 1024$ letters on the board. How many adjacent pairs are the same letter?

11. [**8**] Let $\triangle ABC$ be an acute triangle, with M being the midpoint of $\overline{BC}$, such that $AM = BC$. Let D and E be the intersection of the internal angle bisectors of $\angle AMB$ and $\angle AMC$ with AB and AC, respectively. Find the ratio of the area of $\triangle DME$ to the area of $\triangle ABC$.

12. [**8**] Consider an unusual biased coin, with probability p of landing heads, probability $q \leq p$ of landing tails, and probability $\frac{1}{6}$ of landing on its side (i.e. on neither face). It is known that if this coin is flipped twice, the likelihood that both flips will have the same result is $\frac{1}{2}$. Find p.

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13. [**9**] Find the smallest positive integer n for which

$$1!2!\cdots(n-1)! > n!^2.$$

14. [**9**] Call a triangle *nice* if the plane can be tiled using congruent copies of this triangle so that any two triangles that share an edge (or part of an edge) are reflections of each other via the shared edge. How many dissimilar nice triangles are there?

15. [**9**] On a computer screen is the single character `a`. The computer has two keys: `c` (copy) and `p` (paste), which may be pressed in any sequence.

    Pressing `p` increases the number of `a`'s on screen by the number that were there the last time `c` was pressed. `c` doesn't change the number of `a`'s on screen. Determine the fewest number of keystrokes required to attain at least 2018 `a`'s on screen. (**Note**: pressing `p` before the first press of `c` does nothing).

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16. [**10**] A positive integer is called *primer* if it has a prime number of distinct prime factors. A positive integer is called *primest* if it has a primer number of distinct primer factors. Find the smallest primest number.

17. [**10**] Pascal has a triangle. In the nth row, there are $n+1$ numbers $a_{n,0}, a_{n,1}, a_{n,2}, \ldots, a_{n,n}$ where $a_{n,0} = a_{n,n} = 1$. For all $1 \leq k \leq n-1$, $a_{n,k} = a_{n-1,k} - a_{n-1,k-1}$. What is the sum of the absolute values of all numbers in the 2018th row?

18. [**10**] An $n \times m$ maze is an $n \times m$ grid in which each cell is one of two things: a wall, or a blank. A maze is *solvable* if there exists a sequence of adjacent blank cells from the top left cell to the bottom right cell going through no walls. (In particular, the top left and bottom right cells must both be blank.) Determine the number of solvable 2×5 mazes.

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19. [**11**] Let A be the number of unordered pairs of ordered pairs of integers between 1 and 6 inclusive, and let B be the number of ordered pairs of unordered pairs of integers between 1 and 6 inclusive. (Repetitions are allowed in both ordered and unordered pairs.) Find $A - B$.

20. [**11**] Let z be a complex number. In the complex plane, the distance from z to 1 is 2, and the distance from z^2 to 1 is 6. What is the real part of z?

21. [**11**] A function $f : \{1,2,3,4,5\} \to \{1,2,3,4,5\}$ is said to be *nasty* if there do not exist distinct $a, b \in \{1,2,3,4,5\}$ satisfying $f(a) = b$ and $f(b) = a$. How many nasty functions are there?

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22. [**12**] In a square of side length 4, a point on the interior of the square is randomly chosen and a circle of radius 1 is drawn centered at the point. What is the probability that the circle intersects the square exactly twice?

23. [**12**] Let S be a subset with four elements chosen from $\{1, 2, \ldots, 10\}$. Michael notes that there is a way to label the vertices of a square with elements from S such that no two vertices have the same label, and the labels adjacent to any side of the square differ by at least 4. How many possibilities are there for the subset S?

24. [**12**] Let $ABCD$ be a convex quadrilateral so that all of its sides and diagonals have integer lengths. Given that $\angle ABC = \angle ADC = 90^\circ$, $AB = BD$, and $CD = 41$, find the length of BC.

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25. [**13**] Let $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ be geometric sequences with common ratios r_a and r_b, respectively, such that

$$\sum_{i=0}^{\infty} a_i = \sum_{i=0}^{\infty} b_i = 1 \quad \text{and} \quad \left(\sum_{i=0}^{\infty} a_i^2\right)\left(\sum_{i=0}^{\infty} b_i^2\right) = \sum_{i=0}^{\infty} a_i b_i.$$

Find the smallest real number c such that $a_0 < c$ must be true.

26. [**13**] Points E, F, G, H are chosen on segments AB, BC, CD, DA, respectively, of square $ABCD$. Given that segment EG has length 7, segment FH has length 8, and that EG and FH intersect inside $ABCD$ at an acute angle of 30°, then compute the area of square $ABCD$.

27. [**13**] At lunch, Abby, Bart, Carl, Dana, and Evan share a pizza divided radially into 16 slices. Each one takes takes one slice of pizza uniformly at random, leaving 11 slices. The remaining slices of pizza form "sectors" broken up by the taken slices, e.g. if they take five consecutive slices then there is one sector, but if none of them take adjacent slices then there will be five sectors. What is the expected number of sectors formed?

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28. [**15**] What is the 3-digit number formed by the 9998^{th} through 10000^{th} digits after the decimal point in the decimal expansion of $\frac{1}{998}$?

Note: Make sure your answer has exactly three digits, so please include any leading zeroes if necessary.

29. [**15**] An isosceles right triangle ABC has area 1. Points D, E, F are chosen on BC, CA, AB respectively such that DEF is also an isosceles right triangle. Find the smallest possible area of DEF.

30. [**15**] Let n be a positive integer. Let there be P_n ways for Pretty Penny to make exactly n dollars out of quarters, dimes, nickels, and pennies. Also, let there be B_n ways for Beautiful Bill to make exactly n dollars out of one dollar bills, quarters, dimes, and nickels. As n goes to infinity, the sequence of fractions $\frac{P_n}{B_n}$ approaches a real number c. Find c.

Note: Assume both Pretty Penny and Beautiful Bill each have an unlimited number of each type of coin. Pennies, nickels, dimes, quarters, and dollar bills are worth 1, 5, 10, 25, 100 cents respectively.

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31. [**17**] David and Evan each repeatedly flip a fair coin. David will stop when he flips a tail, and Evan will stop once he flips 2 consecutive tails. Find the probability that David flips more total heads than Evan.

32. [**17**] Over all real numbers x and y, find the minimum possible value of

$$(xy)^2 + (x+7)^2 + (2y+7)^2.$$

33. [**17**] Let ABC be a triangle with $AB = 20, BC = 10, CA = 15$. Let I be the incenter of ABC, and let BI meet AC at E and CI meet AB at F. Suppose that the circumcircles of BIF and CIE meet at a point D different from I. Find the length of the tangent from A to the circumcircle of DEF.

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34. [**20**] A positive integer is called *primer* if it has a prime number of distinct prime factors. A positive integer is called *primest* if it has a primer number of distinct primer factors. A positive integer is called *prime-minister* if it has a primest number of distinct primest factors. Let N be the smallest prime-minister number. Estimate N.

    An estimate of $E > 0$ earns $\lfloor 20 \min\left(\frac{N}{E}, \frac{E}{N}\right) \rfloor$ points.

35. [**20**] Pascal has a triangle. In the nth row, there are $n+1$ numbers $a_{n,0}, a_{n,1}, a_{n,2}, \ldots, a_{n,n}$ where $a_{n,0} = a_{n,n} = 1$. For all $1 \le k \le n-1$, $a_{n,k} = a_{n-1,k} - a_{n-1,k-1}$. Let N be the value of the sum

$$\sum_{k=0}^{2018} \frac{|a_{2018,k}|}{\binom{2018}{k}}.$$

    Estimate N.

    An estimate of $E > 0$ earns $\lfloor 20 \cdot 2^{-|N-E|/70} \rfloor$ points.

36. [**20**] An $n \times m$ maze is an $n \times m$ grid in which each cell is one of two things: a wall, or a blank. A maze is *solvable* if there exists a sequence of adjacent blank cells from the top left cell to the bottom right cell going through no walls. (In particular, the top left and bottom right cells must both be blank.) Let N be the number of solvable 5×5 mazes. Estimate N.

    An estimate of $E > 0$ earns $\lfloor 20 \min\left(\frac{N}{E}, \frac{E}{N}\right)^2 \rfloor$ points.