HMMT November 2018

November 10, 2018

Guts Round

1. [**5**] A positive integer is called *primer* if it has a prime number of distinct prime factors. Find the smallest primer number.

   Proposed by: Yuan Yao

   Answer: $\boxed{6}$

   A primer number must have at least two distinct prime factors, and 6 will work.

2. [**5**] Pascal has a triangle. In the nth row, there are $n+1$ numbers $a_{n,0}, a_{n,1}, a_{n,2}, \ldots, a_{n,n}$ where $a_{n,0} = a_{n,n} = 1$. For all $1 \leq k \leq n-1$, $a_{n,k} = a_{n-1,k} - a_{n-1,k-1}$. What is the sum of all numbers in the 2018th row?

   Proposed by: Michael Ren

   Answer: $\boxed{2}$

   In general, the sum of the numbers on the nth row will be

$$\sum_{k=0}^{n} a_{n,k} = a_{n,0} + \sum_{k=1}^{n-1}(a_{n-1,k} - a_{n-1,k-1}) + a_{n,n} = a_{n,0} + (a_{n-1,n-1} - a_{n-1,0}) + a_{n,n} = 2.$$

3. [**5**] An $n \times m$ maze is an $n \times m$ grid in which each cell is one of two things: a wall, or a blank. A maze is *solvable* if there exists a sequence of adjacent blank cells from the top left cell to the bottom right cell going through no walls. (In particular, the top left and bottom right cells must both be blank.) Determine the number of solvable 2×2 mazes.

   Proposed by: James Lin

   Answer: $\boxed{3}$

   We must have both top-left and bottom-right cells blank, and we cannot have both top-right and bottom-left cells with walls. As long as those conditions are satisfied, the maze is solvable, so the answer is 3.

4. [**6**] Let a, b, c, n be positive real numbers such that $\frac{a+b}{a} = 3$, $\frac{b+c}{b} = 4$, and $\frac{c+a}{c} = n$. Find n.

   Proposed by: Evan Chen

   Answer: $\boxed{\frac{7}{6}}$

   We have

$$1 = \frac{b}{a} \cdot \frac{c}{b} \cdot \frac{a}{c} = (3-1)(4-1)(n-1).$$

   Solving for n yields $n = \frac{7}{6}$.

5. [**6**] Jerry has ten distinguishable coins, each of which currently has heads facing up. He chooses one coin and flips it over, so it now has tails facing up. Then he picks another coin (possibly the same one as before) and flips it over. How many configurations of heads and tails are possible after these two flips?

   Proposed by: Andrew Gu

   Answer: $\boxed{46}$

   We have two cases:

   Case 1: Jerry picks the same coin twice. Then, the first time he flips the coin, it becomes tails, and then the second time, it becomes heads again, giving us the original state of all heads.

   Case 2: Jerry picks two different coins. In this case, there are two coins with tails face up, and the rest are heads face up. There are $\binom{10}{2} = \frac{10 \cdot 9}{2} = 45$ ways to pick which two coins have tails.

   Adding up the possibilities from both cases, we have a total of $1 + 45 = 46$ possible configurations.

6. [**6**] An equilateral hexagon with side length 1 has interior angles $90°, 120°, 150°, 90°, 120°, 150°$ in that order. Find its area.

Proposed by: Yuan Yao

Answer: $\boxed{\frac{3+\sqrt{3}}{2}}$

The area of this hexagon is the area of a $\frac{3}{2} \times \left(1 + \frac{\sqrt{3}}{2}\right)$ rectangle (with the $90°$ angles of the hexagon at opposite vertices) minus the area of an equilateral triangle with side length 1. Then this is

$$\frac{6+3\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = \frac{3+\sqrt{3}}{2}.$$

7. [**7**] At Easter-Egg Academy, each student has two eyes, each of which can be eggshell, cream, or cornsilk. It is known that 30% of the students have at least one eggshell eye, 40% of the students have at least one cream eye, and 50% of the students have at least one cornsilk eye. What percentage of the students at Easter-Egg Academy have two eyes of the same color?

Proposed by: Yuan Yao

Answer: $\boxed{80\%}$

For the purposes of this solution, we abbreviate "eggshell" by "egg", and "cornsilk" by "corn". We know that there are only six combinations of eye color possible: egg-cream, egg-corn, egg-egg, cream-corn, cream-cream, corn-corn. If we let the proportions for each of these be represented by a, b, c, d, e, and f respectively, we have the following four equalities:

$$\begin{aligned} a+b+c &= .3 \\ a+d+e &= .4 \\ b+d+f &= .5 \\ a+b+c+d+e+f &= 1 \end{aligned}$$

where the first three equalities come from the given conditions. Adding the first three equations and subtracting the fourth, we obtain that

$$a+b+d = .2$$

which is the proportion of people with different colored eyes. The proportion of people with the same eye color is thus $1 - .2 = .8$.

8. [**7**] Pentagon $JAMES$ is such that $AM = SJ$ and the internal angles satisfy $\angle J = \angle A = \angle E = 90°$, and $\angle M = \angle S$. Given that there exists a diagonal of $JAMES$ that bisects its area, find the ratio of the shortest side of $JAMES$ to the longest side of $JAMES$.

Proposed by: James Lin

Answer: $\boxed{\frac{1}{4}}$

Since $\angle J = \angle A = 90°$ and $AM = JS$, $JAMS$ must be a rectangle. In addition, $\angle M + \angle S = 270°$, so $\angle M = \angle S = 135°$. Therefore, $\angle ESM = \angle EMS = 45°$, which means MES is an isosceles right triangle. Note that AME and JSE are congruent, which means that $[JAES] = [JAE] + [JSE] = [JAE] + [AME] > [AME]$, so AE cannot be our diagonal. Similarly, JE cannot be our diagonal. Diagonals SA and JM bisect rectangle $JAMS$, so they also cannot bisect the pentagon. Thus, the only diagonal that can bisect $[JAMES]$ is MS, which implies $[JAMS] = [MES]$. We know $[JAMS] = JA \cdot AM$ and $[MES] = \frac{ME \cdot ES}{2}$, and $ME = ES = \frac{JA}{\sqrt{2}}$, which implies

$$JA \cdot AM = \frac{JA^2}{4} \implies \frac{AM}{JA} = \frac{1}{4}$$

Finally, EM and MS are both $\frac{1}{\sqrt{2}}$ the length of $SM = JA$. This means that AM is our shortest side and JA is our longest side, so $\frac{1}{4}$ is our answer.

9. [**7**] Farmer James has some strange animals. His hens have 2 heads and 8 legs, his peacocks have 3 heads and 9 legs, and his zombie hens have 6 heads and 12 legs. Farmer James counts 800 heads and 2018 legs on his farm. What is the number of animals that Farmer James has on his farm?

   Proposed by: James Lin

   Answer: $\boxed{203}$

   Note that each animal has 6 more legs than heads. Thus, if there are n animals, then there are $6n$ more legs than heads in total. There are $2018 - 800 = 1218$ more legs than heads in total, so there are $\frac{1218}{6} = 203$ animals.

10. [**8**] Abbot writes the letter A on the board. Every minute, he replaces every occurrence of A with AB and every occurrence of B with BA, hence creating a string that is twice as long. After 10 minutes, there are $2^{10} = 1024$ letters on the board. How many adjacent pairs are the same letter?

   Proposed by: Yuan Yao

   Answer: $\boxed{341}$

   Let a_n denote the number of adjacent pairs of letters that are the same after n minutes, and b_n the number of adjacent pairs that are different.

   Lemma 1. $a_n = b_{n-1}$ for all $n \geq 0$.

   Proof. Any adjacent pair of identical letters XX at stage n either came from the same letter of stage $n-1$ ($W \to XX$), or two adjacent letters of stage $n-1$ ($VW \to MXXN$). Because $A \to AB$ and $B \to BA$, they cannot have come from the same letter.

   If they came from a pair of adjacent letters, then observing what each adjacent pair of letters results in in the next minute,

$$
\begin{aligned}
AA &\to ABAB \\
AB &\to ABBA \\
BA &\to BAAB \\
BB &\to BABA
\end{aligned}
$$

   we see that our adjacent pair VW must have been AB or BA. The number of such pairs is precisely b_{n-1}.

   From the relation $a_n + b_n = 2^n - 1$ for all $n \geq 0$, we obtain the recurrence relation

$$a_n = 2^{n-1} - 1 - a_{n-1}$$

   from which we obtain values $a_0 = 0$, $a_1 = 0$, $a_2 = 1$, $a_3 = 2$, $a_4 = 5$, $a_5 = 10$, $a_6 = 21$, $a_7 = 42$, $a_8 = 85$, $a_9 = 170$, and $a_{10} = \boxed{341}$.

11. [**8**] Let $\triangle ABC$ be an acute triangle, with M being the midpoint of $\overline{BC}$, such that $AM = BC$. Let D and E be the intersection of the internal angle bisectors of $\angle AMB$ and $\angle AMC$ with AB and AC, respectively. Find the ratio of the area of $\triangle DME$ to the area of $\triangle ABC$.

   Proposed by: Handong Wang

   Answer: $\boxed{\frac{2}{9}}$

   Let $[XYZ]$ denote the area of $\triangle XYZ$.

   Solution 1: Let $AM = \ell$, let $DE = d$, and let the midpoint of $\overline{DE}$ be F. Since $\frac{AD}{AB} = \frac{AE}{AC} = \frac{2}{3}$ by the angle bisector theorem, F lies on $\overline{AM}$ and $\triangle ADE$ is similar to $\triangle ABC$. Note that $\angle DME$ is formed by angle bisectors of $\angle AMB$ and $\angle AMC$, which add up to $180°$. Thus $\angle DME$ is right, so both $\triangle DMF$ and $\triangle EMF$ are isosceles. This implies that $FM = \frac{d}{2}$. Applying the similarity between $\triangle ADE$ and $\triangle ABC$, we get $\frac{d}{\ell} = \frac{AD}{AB} = \frac{2}{3}$. Since $AF = 2FM$, $[ADE] = 2[DME]$. Finally, since $\frac{[ADE]}{[ABC]} = \frac{4}{9}$, $\frac{[DME]}{[ABC]} = \frac{1}{2} \cdot \frac{4}{9} = \boxed{\frac{2}{9}}$.

Solution 2: We compute the ratio $\frac{[DME]}{[ABC]}$ by finding $1 - \frac{[ADE]}{[ABC]} - \frac{[DBM]}{[ABC]} - \frac{[EMC]}{[ABC]}$. Since $\frac{AD}{DB} = \frac{AE}{EC} = \frac{AM}{\frac{1}{2}BC} = 2$ by the angle bisector theorem, we see that $\frac{[ADE]}{[ABC]} = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{4}{9}$. Also, since $BM = CM = \frac{1}{2}BC$, $\frac{[DBM]}{[ABC]} = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{6}$, and $\frac{[EMC]}{[ABC]} = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{1}{6}$. Thus, $\frac{[DME]}{[ABC]} = 1 - \frac{[ADE]}{[ABC]} - \frac{[DBM]}{[ABC]} - \frac{[EMC]}{[ABC]} = 1 - \frac{4}{9} - \frac{1}{6} - \frac{1}{6} = \boxed{\frac{2}{9}}$.

12. [**8**] Consider an unusual biased coin, with probability p of landing heads, probability $q \leq p$ of landing tails, and probability $\frac{1}{6}$ of landing on its side (i.e. on neither face). It is known that if this coin is flipped twice, the likelihood that both flips will have the same result is $\frac{1}{2}$. Find p.

Proposed by: Brian Reinhart

Answer: $\boxed{\frac{2}{3}}$

The probability that both flips are the same is $p^2 + q^2 + \frac{1}{36}$. For this to be $\frac{1}{2}$, we must have

$$p^2 + q^2 + \frac{1}{36} = p^2 + \left(\frac{5}{6} - p\right)^2 + \frac{1}{36} = \frac{1}{2}$$

Using the quadratic formula, $p = \frac{2}{3}$ or $\frac{1}{6}$. Since $p \geq q$, we have that $p = \frac{2}{3}$.

13. [**9**] Find the smallest positive integer n for which

$$1!2! \cdots (n-1)! > n!^2.$$

Proposed by: Andrew Gu

Answer: $\boxed{8}$

Dividing both sides by $n!^2$, we obtain

$$\begin{aligned} \frac{1!2!...(n-3)!(n-2)!(n-1)!}{[n(n-1)!]\,[n(n-1)(n-2)!]} &> 1 \\ \frac{1!2!...(n-3)!}{n^2(n-1)} &> 1 \\ 1!2!...(n-3)! &> n^2(n-1) \end{aligned}$$

Factorials are small at first, so we can rule out some small cases: when $n = 6$, the left hand side is $1!2!3! = 12$, which is much smaller than $6^2 \cdot 5$. (Similar calculations show that $n = 1$ through $n = 5$ do not work. either.)

Setting $n = 7$, the left-hand side is 288, which is still smaller than $7^2 \cdot 6$. However, $n = 8$ gives $34560 > 448$, so 8 is the smallest integer for which the inequality holds.

14. [**9**] Call a triangle *nice* if the plane can be tiled using congruent copies of this triangle so that any two triangles that share an edge (or part of an edge) are reflections of each other via the shared edge. How many dissimilar nice triangles are there?

Proposed by: Yuan Yao

Answer: $\boxed{4}$

The triangles are 60-60-60, 45-45-90, 30-60-90, and 30-30-120.

We make two observations.

- By reflecting ”around” the same point, any angle of the triangle must be an integer divisor of $360°$.
- if any angle is an odd divisor of $360°$, i.e equals $\frac{360}{k}$ for odd k, then the two adjacent sides must be equal.

We do casework on the largest angle.

- $60°$. We are forced into a $\boxed{\text{60-60-60}}$ triangle, which works.
- $72°$. By observation 2, this triangle's other two angles are $54°$. This is not an integer divisor of $360°$.
- $90°$. The second largest angle is at least $45°$. If it is $45°$, it is the valid $\boxed{\text{90-45-45}}$ triangle. If it is $\frac{360°}{7}$, the triangle is invalid by observation 2. If it is $60°$, it is the valid $\boxed{\text{90-60-30}}$ triangle. If it is $72°$, the triangle is invalid by observation 2.
- $120°$. By observation 2, the other angles are $30°$, meaning it is the valid $\boxed{\text{120-30-30}}$ triangle.

The conclusion follows.

15. [**9**] On a computer screen is the single character `a`. The computer has two keys: `c` (copy) and `p` (paste), which may be pressed in any sequence.

Pressing `p` increases the number of `a`'s on screen by the number that were there the last time `c` was pressed. `c` doesn't change the number of `a`'s on screen. Determine the fewest number of keystrokes required to attain at least 2018 `a`'s on screen. (**Note**: pressing `p` before the first press of `c` does nothing).

Proposed by: John Michael Wu

Answer: $\boxed{21}$

The first keystroke must be `c` and the last keystroke must be `p`. If there are k `c`'s pressed in total, let n_i denote one more than the number of `p`'s pressed immediately following the i'th `c`, for $1 \leq i \leq k$.

Then, we have that the total number of keystrokes is

$$s := \sum_{i=1}^{k} n_i$$

and the total number of `a`'s is

$$r := \prod_{i=1}^{k} n_i$$

We desire to minimize s with the constraint that $r \geq 2018$. We claim that the minimum possible s is $s = 21$.

This value of s is achieved by $k = 7$ and $n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = n_7 = 3$, so it remains to show that $s = 20$ is not possible.

Suppose it were for some k and n_i. By the AM-GM inequality,

$$\left(\frac{n_1 + n_2 + \cdots + n_k}{k}\right) \geq \sqrt[k]{n_1 n_2 \cdots n_k}$$

implying that

$$\begin{aligned} 2018 &\leq n_1 n_2 \cdots n_k \\ &\leq \left(\frac{n_1 + n_2 + \cdots + n_k}{k}\right)^k \\ &= \left(\frac{20}{k}\right)^k \end{aligned}$$

which is satisfied by no positive integers k. More rigorously, the function $f(x) = x^{\frac{1}{x}}$ is well known to have a maximum at $x = e$. Making the substitution $u = \frac{20}{k}$, we obtain

$$\begin{aligned} \left(\frac{20}{k}\right)^k &= u^{\frac{20}{u}} \\ &= \left(u^{\frac{1}{u}}\right)^{20} \end{aligned}$$

which is maximized by setting $u = e$. However, $e^{\frac{20}{e}} \approx 1568.05$, meaning that $s = 20$ is not possible.

16. [**10**] A positive integer is called *primer* if it has a prime number of distinct prime factors. A positive integer is called *primest* if it has a primer number of distinct primer factors. Find the smallest primest number.

Proposed by: Yuan Yao

Answer: $\boxed{72}$

We claim the answer is 72, as it has 6 primer factors: $6, 12, 24, 18, 36, 72$, and 6 is a primer.

We now prove that there is no smaller primer. Suppose there were a smaller primer $r < 72$. We do casework on the number of distinct prime factors of r.

- r has ≥ 4 distinct prime factors. Then $r \geq 2 \cdot 3 \cdot 5 \cdot 7 = 210$, which is larger than 72.
- r has 3 distinct prime factors. If each of these factors has multiplicity 1, i.e $r = pqs$ for distinct primes p, q, s, then r has precisely 4 primer factors: pq, qs, sp, pqs, and 4 is not a primer. Thus, r contains at least one factor of multiplicity at least 2. If r is p^2qs for distinct primes p, q, s, then r has 7 distinct primer factors: $pq, qs, sp, pqs, p^2q, sp^2, p^2qs$, and 7 is not a primer. Thus, if $r = p^a q^b s^c$, $a + b + c \geq 5$, and $r \geq 2^3 \cdot 3 \cdot 5 = 120$, which is ≥ 72.
- r has 2 distinct prime factors. If $r = p^a q^b$, for distinct primts p, q, then r's primer factors are precisely its divisors of the form $p^i q^j$, where $1 \leq i \leq a$ and $1 \leq j \leq b$, meaning that it has ab primer factors. Thus, ab is a primer, meaning that $ab \geq 6$. Thus $r \geq 2^3 \cdot 3^2 = 72$, where the other possibilities can be ruled out through easy casework.
- r has 1 distinct prime factor. Then it doesn't have any primer factors, and thus cannot possibly have a primer number of them.

We conclude that 72 is the smallest primer number.

17. [**10**] Pascal has a triangle. In the nth row, there are $n + 1$ numbers $a_{n,0}, a_{n,1}, a_{n,2}, \ldots, a_{n,n}$ where $a_{n,0} = a_{n,n} = 1$. For all $1 \leq k \leq n - 1$, $a_{n,k} = a_{n-1,k} - a_{n-1,k-1}$. What is the sum of the absolute values of all numbers in the 2018th row?

Proposed by: Michael Ren

Answer: $\boxed{\frac{2^{2018}+2}{3}}$

Let s_n be the sum of the absolute values of numbers in the nth row. For odd n, we have that $a_{n,1}, \ldots, a_{n,n-1}$ alternate in sign as $-, +, -, +, \ldots, +$, with the last term being $a_{n,n-1} = 1$. For even

n, we have that $a_{n,1}, \ldots, a_{n,n-2}$ alternate in sign as $-, +, -, +, \ldots, +$, and $a_{n,n-1} = 0$. These facts can be proven by induction. Thus, $s_n = 1 - a_{n,1} + a_{n,2} - \cdots + (-1)^{n-1}a_{n,n-1} + 1$. Applying the recursion, for $n > 0$ this becomes $s_n = 1 - (a_{n-1,1} - a_{n-1,0}) + (a_{n-1,2} - a_{n-1,1}) - \cdots + (-1)^{n-1}(a_{n-1,n-1} - a_{n-1,n-2}) + 1 = 2(1 - a_{n-1,1} + a_{n-1,2} - \cdots + (-1)^{n-2}a_{n-1,n-2} + 1) - 1 + (-1)^{n-1}$. In other words, if n is even then $s_n = 2s_{n-1} - 2$ and if n is odd then $s_n = 2s_{n-1}$. This means that $s_{2n} = 4s_{2n-2} - 2$. Since 2018 is even, we can write $s_{2018} = 4s_{2016} - 2 = 2^{2018} - 2^{2017} - 2^{2015} - \cdots - 2$. Applying the formula for the sum of a geometric series, we get $s_{2018} = 2^{2018} - \frac{2^{2019}-2}{4-1} = \frac{2^{2018}+2}{3}$.

18. [**10**] An $n \times m$ maze is an $n \times m$ grid in which each cell is one of two things: a wall, or a blank. A maze is *solvable* if there exists a sequence of adjacent blank cells from the top left cell to the bottom right cell going through no walls. (In particular, the top left and bottom right cells must both be blank.) Determine the number of solvable 2×5 mazes.

Proposed by: John Michael Wu

Answer: $\boxed{49}$

Solution 1: Replace 5 by an arbitrary n. Label the cells of the maze by (x, y) where $1 \leq x \leq n$ and $1 \leq y \leq 2$. Let a_n denote the number of solvable $2 \times n$ mazes, and let b_n denote the number of $2 \times n$ mazes where there exists a sequence of adjacent blank cells from the $(1, 1)$ to $(n, 1)$. We observe the relations

$$a_n = 2a_{n-1} + b_{n-2}$$
$$b_n = 2b_{n-1} + a_{n-2}.$$

The first relation follows from dividing into cases depending on if $(n-1, 2$ is blank. If this cell is blank, then the maze is solvable if and only if there is a path to $(n-1, 2)$ and the cell $(n, 2)$ is blank. The cell $(n, 1)$ is arbitrary so we get $2a_{n-1}$. If $(n-1, 2)$ is a wall, then the maze is solvable if and only if there is a path to $(n-2, 1)$ and each of the cells $(n-1, 1), (n, 1), (n, 2)$ are blank. This gives the term b_{n-2}.

The second relation follows similarly dividing into cases based on whether the cell $(n-1, 1)$ is blank or not.

The base cases are $a_1 = 1, a_2 = 3$, $b_1 = 2, b_2 = 4$. We thus obtain:

n	a_n	b_n
1	1	2
2	3	4
3	8	9
4	20	21
5	49	50

The answer is then 49.

Solution 2: Call a maze *reachable* if there exists a sequence of adjacent blank cells from $(1, 1)$ to any cell in column n. Let x_n denote the number of reachable $2 \times n$ mazes where the rightmost column has a blank in the top cell and a wall in the bottom cell, let y_n denote the number of reachable $2 \times n$ mazes where the rightmost column has a wall in the top cell and a blank in the bottom cell, and let z_n denote the number of reachable $2 \times n$ mazes where the rightmost column has a blank in both of its cells. Then, observe the relations

$$x_n = z_{n-1} + x_{n-1}$$
$$y_n = z_{n-1} + y_{n-1}$$
$$z_n = z_{n-1} + x_{n-1} + y_{n-1},$$

as well as base cases $x_1 = 1, y_1 = 0, z_1 = 1$. We thus obtain:

n	x_n	y_n	z_n
1	1	0	1
2	2	1	2
3	4	3	5
4	9	8	12
5	21	20	29

Note that the answer is $y_5 + z_5$, which is $20 + 29 = 49$.

19. [**11**] Let A be the number of unordered pairs of ordered pairs of integers between 1 and 6 inclusive, and let B be the number of ordered pairs of unordered pairs of integers between 1 and 6 inclusive. (Repetitions are allowed in both ordered and unordered pairs.) Find $A - B$.

Proposed by: Yuan Yao

Answer: $\boxed{225}$

There are $6 \cdot 6$ ordered pairs of integers between 1 and 6 inclusive and 21 unordered pairs of integers ($\binom{6}{2} = 15$ different pairs and 6 doubles). Then, $A = \binom{36}{2} + 36 = 666$ and $B = 21 \cdot 21 = 441$. Therefore $A - B = 225$.

For general n, there are n^2 ordered pairs of integers and $\frac{n(n+1)}{2}$ unordered pairs of integers. Then $A = \frac{n^2(n^2+1)}{2}$ and $B = \frac{n^2(n+1)^2}{4}$ so

$$A - B = \frac{n^2(2(n^2+1) - (n+1)^2)}{4} = \left(\frac{n(n-1)}{2}\right)^2.$$

20. [**11**] Let z be a complex number. In the complex plane, the distance from z to 1 is 2, and the distance from z^2 to 1 is 6. What is the real part of z?

Proposed by: James Lin

Answer: $\boxed{\frac{5}{4}}$

Note that we must have $|z-1| = 2$ and $|z^2 - 1| = 6$, so $|z+1| = \frac{|z^2-1|}{|z-1|} = 3$. Thus, the distance from z to 1 in the complex plane is 2 and the distance from z to -1 in the complex plane is 3. Thus, $z, 1, -1$ form a triangle with side lengths $2, 3, 3$. The area of a triangle with sides $2, 2, 3$ can be computed to be $\frac{3\sqrt{7}}{4}$ by standard techniques, so the length of the altitude from z to the real axis is $\frac{3\sqrt{7}}{4} \cdot \frac{2}{2} = \frac{3\sqrt{7}}{4}$. The distance between 1 and the foot from z to the real axis is $\sqrt{2^2 - \left(\frac{3\sqrt{7}}{4}\right)^2} = \frac{1}{4}$ by the Pythagorean Theorem. It is clear that z has positive imaginary part as the distance from z to -1 is greater than the distance from z to 1, so the distance from 0 to the foot from z to the real axis is $1 + \frac{1}{4} = \frac{5}{4}$. This is exactly the real part of z that we are trying to compute.

21. [**11**] A function $f : \{1,2,3,4,5\} \to \{1,2,3,4,5\}$ is said to be *nasty* if there do not exist distinct $a, b \in \{1,2,3,4,5\}$ satisfying $f(a) = b$ and $f(b) = a$. How many nasty functions are there?

Proposed by: Michael Ren

Answer: $\boxed{1950}$

We use complementary counting. There are $5^5 = 3125$ total functions. If there is at least one pair of numbers which map to each other, there are $\binom{5}{2} = 10$ ways to choose the pair and $5^3 = 125$ ways to assign the other values of the function for a total of 1250. But we overcount each time there are two such pairs, so we must subtract off $5 \cdot 3 \cdot 5 = 75$, where there are 5 options for which number is not in a pair, 3 options for how the other four numbers are paired up, and 5 options for where the function outputs when the unpaired number is inputted. This results in a final answer of $3125 - (1250 - 75) = 1950$.

22. [**12**] In a square of side length 4, a point on the interior of the square is randomly chosen and a circle of radius 1 is drawn centered at the point. What is the probability that the circle intersects the square exactly twice?

Proposed by: Jason Lu

Answer: $\boxed{\frac{\pi+8}{16}}$

Consider the two intersection points of the circle and the square, which are either on the same side of the square or adjacent sides of the square. In order for the circle to intersect a side of the square twice, it must be at distance at most 1 from that side and at least 1 from all other sides. The region of points where the center could be forms a 2×1 rectangle.

In the other case, a square intersects a pair of adjacent sides once each if it it at distance at most one from the corner, so that the circle contains the corner. The region of points where the center could be is a quarter-circle of radius 1.

The total area of the regions where the center could be is $\pi + 8$, so the probability is $\frac{\pi+8}{16}$.

23. [**12**] Let S be a subset with four elements chosen from $\{1, 2, \ldots, 10\}$. Michael notes that there is a way to label the vertices of a square with elements from S such that no two vertices have the same label, and the labels adjacent to any side of the square differ by at least 4. How many possibilities are there for the subset S?

Proposed by: James Lin

Answer: $\boxed{36}$

Let the four numbers be a, b, c, d around the square. Assume without loss of generality that a is the largest number, so that $a > b$ and $a > d$. Note that c cannot be simultaneously smaller than one of b, d and larger than the other because, e.g. if $b > c > d$, then $a > b > c > d$ and $a \geq d + 12$. Hence c is either smaller than b and d or larger than b and d.

Case 1: c is smaller than b and d. Then we have $a - c \geq 8$, but when $a - c = 8$, we have $b = c + 4 = d$, so we need $a - c = 9$, giving the only set $\{1, 5, 6, 10\}$.

Case 2: c is larger than b and d. Since $a > c$ and b, d are both at most $c - 4$, the range of possible values for c is $\{6, 7, 8, 9\}$. When $c = 9, 8, 7, 6$, there are $1, 2, 3, 4$ choices for a respectively and $\binom{5}{2}, \binom{4}{2}, \binom{3}{2}, \binom{2}{2}$ for b and d respectively (remember that order of b and d does not matter). So there are $1 \cdot 10 + 2 \cdot 6 + 3 \cdot 3 + 4 \cdot 1 = 35$ sets in this case.

Therefore we have $1 + 35 = 36$ possible sets in total.

24. [**12**] Let $ABCD$ be a convex quadrilateral so that all of its sides and diagonals have integer lengths. Given that $\angle ABC = \angle ADC = 90^\circ$, $AB = BD$, and $CD = 41$, find the length of BC.

Proposed by: Anders Olsen

Answer: $\boxed{580}$

Let the midpoint of AC be O which is the center of the circumcircle of $ABCD$. ADC is a right triangle with a leg of length 41, and $41^2 = AC^2 - AD^2 = (AC - AD)(AC + AD)$. As AC, AD are integers and 41 is prime, we must have $AC = 840, AD = 841$. Let M be the midpoint of AD. $\triangle AOM \sim \triangle ACD$, so $BM = BO + OM = 841/2 + 41/2 = 441$. Then $AB = \sqrt{420^2 + 441^2} = 609$ (this is a 20-21-29 triangle scaled up by a factor of 21). Finally, $BC^2 = AC^2 - AB^2$ so $BC = \sqrt{841^2 - 609^2} = 580$.

25. [**13**] Let $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ be geometric sequences with common ratios r_a and r_b, respectively, such that

$$\sum_{i=0}^{\infty} a_i = \sum_{i=0}^{\infty} b_i = 1 \quad \text{and} \quad \left(\sum_{i=0}^{\infty} a_i^2\right)\left(\sum_{i=0}^{\infty} b_i^2\right) = \sum_{i=0}^{\infty} a_i b_i.$$

Find the smallest real number c such that $a_0 < c$ must be true.

Proposed by: Handong Wang

Answer: $\boxed{\frac{4}{3}}$

Let $a_0 = a$ and $b_0 = b$. From $\sum_{i=0}^{\infty} a_i = \frac{a_0}{1-r_a} = 1$ we have $a_0 = 1 - r_a$ and similarly $b_0 = 1 - r_b$. This means $\sum_{i=0}^{\infty} a_i^2 = \frac{a_0^2}{1-r_a^2} = \frac{a^2}{(1-r_a)(1+r_a)} = \frac{a^2}{a(2-a)} = \frac{a}{2-a}$, so $\sum_{i=0}^{\infty} a_i^2 \sum_{i=0}^{\infty} b_i^2 = \sum_{i=0}^{\infty} a_i b_i$ yields

$$\frac{a}{2-a} \cdot \frac{b}{2-b} = \frac{ab}{1-(1-a)(1-b)}.$$

Since the numerators are equal, the denominators must be equal, which when expanded gives $2ab - 3a - 3b + 4 = 0$, which is equivalent to $(2a-3)(2b-3) = 1$. But note that $0 < a, b < 2$ since we need the sequences to converge (or $|r_a|, |r_b| < 1$), so then $-3 < 2b - 3 < 1$, and thus $2a - 3 > 1$ (impossible) or $2a - 3 < -\frac{1}{3}$. Hence $a < \frac{4}{3}$, with equality when b approaches 0.

26. [**13**] Points E, F, G, H are chosen on segments AB, BC, CD, DA, respectively, of square $ABCD$. Given that segment EG has length 7, segment FH has length 8, and that EG and FH intersect inside $ABCD$ at an acute angle of $30°$, then compute the area of square $ABCD$.

Proposed by: Kevin Sun

Answer: $\boxed{\frac{784}{19}}$

Rotate EG by $90°$ about the center of the square to $E'G'$ with $E' \in AD$ and $G' \in BC$. Now $E'G'$ and FH intersect at an angle of $60°$. Then consider the translation which takes E' to H and G' to I. Triangle FHI has $FH = 8, HI = 7$ and $\angle FHI = 60°$. Furthermore, the height of this triangle is the side length of the square. Using the Law of Cosines,

$$FI = \sqrt{7^2 - 7 \cdot 8 + 8^2} = \sqrt{57}.$$

By computing the area of FHI in two ways, if h is the height then

$$\frac{1}{2} \times \sqrt{57} \times h = \frac{1}{2} \times \frac{\sqrt{3}}{2} \times 7 \times 8.$$

Then $h = \frac{28}{\sqrt{19}}$ and the area of the square is $h^2 = \frac{784}{19}$.

27. [**13**] At lunch, Abby, Bart, Carl, Dana, and Evan share a pizza divided radially into 16 slices. Each one takes takes one slice of pizza uniformly at random, leaving 11 slices. The remaining slices of pizza form "sectors" broken up by the taken slices, e.g. if they take five consecutive slices then there is one sector, but if none of them take adjacent slices then there will be five sectors. What is the expected number of sectors formed?

Proposed by: Andrew Gu

Answer: $\boxed{\frac{11}{3}}$

Consider the more general case where there are N slices and $M > 0$ slices are taken. Let S denote the number of adjacent pairs of slices of pizza which still remain. There are $N - M$ slices and a sector of k slices contributes $k - 1$ pairs to S. Hence the number of sectors is $N - M - S$. We compute the expected value of S by looking at each adjacent pair in the original pizza:

$$\mathbb{E}(S) = N \frac{\binom{N-2}{M}}{\binom{N}{M}} = N \frac{(N-M)(N-M-1)}{N(N-1)} = \frac{(N-M)(N-M-1)}{N-1}$$

The expected number of sectors is then

$$N - M - \frac{(N-M)(N-M-1)}{N-1} = \frac{(N-M)M}{N-1}.$$

For $N = 16, M = 5$ this yields $\frac{11}{3}$.

28. [**15**] What is the 3-digit number formed by the 9998^{th} through 10000^{th} digits after the decimal point in the decimal expansion of $\frac{1}{998}$?

Note: Make sure your answer has exactly three digits, so please include any leading zeroes if necessary.

Proposed by: Sujay Kazi

Answer: $\boxed{042}$

Note that $\frac{1}{998} + \frac{1}{2} = \frac{250}{499}$ repeats every 498 digits because 499 is prime, so $\frac{1}{998}$ does as well (after the first 498 block). Now we need to find 38^{th} to 40^{th} digits. We expand this as a geometric series

$$\frac{1}{998} = \frac{\frac{1}{1000}}{1 - \frac{2}{1000}} = .001 + .001 \times .002 + .001 \times .002^2 + \cdots .$$

The contribution to the 36^{th} through 39^{th} digits is 4096, the 39^{th} through 42^{nd} digits is 8192, and 41^{st} through 45^{th} digits is 16384. We add these together:

$$\begin{array}{ccccccccc} 4 & 0 & 9 & 6 & & & & & \\ & & & 8 & 1 & 9 & 2 & & \\ & & & & & 1 & 6 & 8 & 3 \quad 4 \\ \hline 4 & 1 & 0 & 4 & 2 & 0 & \cdots & & \end{array}$$

The remaining terms decrease too fast to have effect on the digits we are looking at, so the 38^{th} to 40^{th} digits are 042.

29. [**15**] An isosceles right triangle ABC has area 1. Points D, E, F are chosen on BC, CA, AB respectively such that DEF is also an isosceles right triangle. Find the smallest possible area of DEF.

Proposed by: Yuan Yao

Answer: $\boxed{\frac{1}{5}}$

Without loss of generality, suppose that AB is the hypotenuse.

If F is the right angle, then F must be the midpoint of AB. To prove this, let X and Y be the feet from F to BC and AC. Since $\angle XFY = \angle DFE = 90°$, we have $\angle XFD = \angle YFE$ so

$$XF = DF \cos \angle XFD = EF \cos \angle YFE = YF.$$

Hence F is equidistant from AC and BC so it is the midpoint of AB. Then the minimum area is achieved by minimizing DF; this occurs when DF is perpendicular to BC. The triangle DEF then becomes the medial triangle of ABC, so its area is $\frac{1}{4}$.

If F is not the right angle, without loss of generality, let the right angle be D. Place this triangle in the complex plane such that C is the origin, $B = \sqrt{2}$, and $A = \sqrt{2}i$.

Now since D is on the real axis and E is on the imaginary axis, $D = x$ and $E = yi$, and we can obtain F by a 90 degree counterclockwise rotation of D around E: this evaluates to $F = y + (x + y)i$. For F to be on AB, the only constraint is to have $y + (x + y) = \sqrt{2} \implies x = \sqrt{2} - 2y$.

To minimize the area, we minimize

$$\frac{DE^2}{2} = \frac{x^2 + y^2}{2} = \frac{(\sqrt{2} - 2y)^2 + y^2}{2} = \frac{5y^2 - 4\sqrt{2}y + 2}{2}$$

which has a minimum of $\frac{1}{5}$ at $y = \frac{2\sqrt{2}}{5}$. Since this is between 0 and $\sqrt{2}$, this is indeed a valid configuration.

Finally, we take the smallest area of $\frac{1}{5}$.

30. [**15**] Let n be a positive integer. Let there be P_n ways for Pretty Penny to make exactly n dollars out of quarters, dimes, nickels, and pennies. Also, let there be B_n ways for Beautiful Bill to make exactly n dollars out of one dollar bills, quarters, dimes, and nickels. As n goes to infinity, the sequence of fractions $\frac{P_n}{B_n}$ approaches a real number c. Find c.

Note: Assume both Pretty Penny and Beautiful Bill each have an unlimited number of each type of coin. Pennies, nickels, dimes, quarters, and dollar bills are worth 1, 5, 10, 25, 100 cents respectively.

Proposed by: James Lin

Answer: $\boxed{20}$

Let d_x be the number ways to make exactly x cents using only dimes and nickels. It is easy to see that when x is a multiple of 5,

$$d_x = \left\lfloor \frac{x}{10} \right\rfloor + 1.$$

Now, let c_x be the number of ways to make exactly x cents using only quarters, dimes and nickels. Again, it is easy to see that when x is a multiple of 5,

$$c_x = c_{x-25} + d_x.$$

(We can either use 1 or more quarters, which corresponds to the c_{x-25} term, or we can use 0 quarters, which corresponds to the d_x term.) Combining these two equations, we see that c_x can be approximated by a polynomial of degree 2. (In fact, we get five different approximations of c_x, depending on the value of $x \pmod{25}$, but they all only differ by a constant, which will not affect the limit case.) We also see that

$$B_n = c_{100n} + c_{100(n-1)} + \ldots + c_0$$

and

$$P_n = c_{100n} + c_{100n-5} + \ldots + c_0.$$

Suppose a is the value such that $\lim_{n\to\infty} \frac{c_n}{an^2} = 1$. Then

$$\lim_{n\to\infty} \frac{B_n}{P_n} = \lim_{n\to\infty} \frac{\sum_{k=0}^{\lfloor n/100 \rfloor} a(100k)^2}{\sum_{k=0}^{\lfloor n/5 \rfloor} a(5k)^2} = \lim_{n\to\infty} \frac{400 \cdot \frac{n}{100}(\frac{n}{100}+1)(2\cdot\frac{n}{100}+1)}{\frac{n}{5}(\frac{n}{5}+1)(2\cdot\frac{n}{5}+1)} = 20.$$

31. [**17**] David and Evan each repeatedly flip a fair coin. David will stop when he flips a tail, and Evan will stop once he flips 2 consecutive tails. Find the probability that David flips more total heads than Evan.

Proposed by: Anders Olsen

Answer: $\boxed{\frac{1}{5}}$

Solution 1: We can find the values of the functions $D(h)$ and $E(h)$, the probabilities that David and Evan, respectively, flip exactly h heads. It is easy to see that $D(h) = 2^{-h-1}$. In order to find $E(h)$, we note that each sequence must end with the flips HTT (unless Evan flips only 2 heads). We disregard these flips for now. Then there are h prior places we can include an extra tails in the sequence, one between each pair of heads. There is a 2^{-h+1} probability of this happening with no extra tails, $h2^{-h}$ probability with 1 extra tail, $\binom{h}{2}2^{-h-1}$ probability with 2 extra tails, and so on. This sum is

$$2^{-h+1}\sum_{n=0}^{h} 2^{-n}\binom{h}{n} = 2\left(\frac{3}{4}\right)^h.$$

We divide by 8 to account for the probability of getting HTT to finish our sequence to get that

$$E(h) = \frac{3^h}{4^{h+1}}.$$

Our answer is

$$\sum_{n=0}^{\infty}\left(E(n)\sum_{m=n+1}^{\infty}D(m)\right)=\sum_{n=0}^{\infty}\frac{3^n}{8^{n+1}}=\frac{1}{5}.$$

Solution 2: Since we only care about the number of heads, we think of this as a "survival" game where they flip a single head each round, such that David has a $\frac{1}{2}$ chance of flipping another head and Evan has a $\frac{3}{4}$ chance of flipping another head. (If they don't get to flip another head, they lose.) David wins if and only if when at least one of David and Evan loses, David does not lose but Evan loses. The probability that at least one of them lose each round is $1-\frac{1}{2}\cdot\frac{3}{4}=\frac{5}{8}$, and David wins this round with probability $\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{8}$, so the overall probability is $\frac{1}{5}$.

32. [**17**] Over all real numbers x and y, find the minimum possible value of

$$(xy)^2+(x+7)^2+(2y+7)^2.$$

Proposed by: James Lin

Answer: $\boxed{45}$

Solution 1: Rewrite the given expression as $(x^2+4)(1+y^2)+14(x+2y)+94$. By Cauchy-Schwartz, this is at least $(x+2y)^2+14(x+2y)+94=(x+2y+7)^2+45$. The minimum is 45, attained when $xy=2, x+2y=-7$.

Solution 2: Let $z=2y, s=x+z, p=xz$. We seek to minimize

$$\begin{aligned}\left(\frac{xz}{2}\right)^2+(x+7)^2+(z+7)^2&=\frac{p^2}{4}+(x^2+z^2)+14(x+z)+98\\&=\frac{p^2}{4}+s^2-2p+14s+98\\&=\left(\frac{p}{2}-2\right)^2+(s+7)^2+45\\&\geq 45.\end{aligned}$$

Equality holds when $s=-7, p=4$. Since $s^2\geq 4p$, this system has a real solution for x and z.

33. [**17**] Let ABC be a triangle with $AB=20, BC=10, CA=15$. Let I be the incenter of ABC, and let BI meet AC at E and CI meet AB at F. Suppose that the circumcircles of BIF and CIE meet at a point D different from I. Find the length of the tangent from A to the circumcircle of DEF.

Proposed by: Michael Ren

Answer: $\boxed{2\sqrt{30}}$

Solution 1: Let $O=AI\cap(AEF)$. We claim that O is the circumcenter of DEF. Indeed, note that $\angle EDF=\angle ECI+\angle FBI=\frac{\angle B+\angle C}{2}=\frac{\angle EOF}{2}$, and $OE=OF$, so the claim is proven.

Now note that the circumcircle of DEF passes through the incenter of AEF, so power of A with respect to (DEF) is $\sqrt{AE\cdot AF}$. We can compute that $AE=10, AF=12$ by the angle bisector theorem, so the length of the tangent is $\sqrt{10\cdot 12}=2\sqrt{30}$.

Solution 2: Let E' be the reflection of E across AI. Note that E' lies on AB. We claim that E' lies on the circumcircle of DEF, which will imply that the power of A with respect to (DEF) is $AE'\cdot AF=AE\cdot AF$ and we proceed as in Solution 1. We can easily compute

$$\angle EE'F=90^\circ+\frac{\angle A}{2}$$

and

$$\angle EDF=\angle EDI+\angle IDF=\angle ECI+\angle IBF=90^\circ-\frac{\angle A}{2}.$$

Therefore $\angle EDF+\angle EE'F=180^\circ$, so E' lies on the circumcircle of DEF as desired.

34. [**20**] A positive integer is called *primer* if it has a prime number of distinct prime factors. A positive integer is called *primest* if it has a primer number of distinct primer factors. A positive integer is called *prime-minister* if it has a primest number of distinct primest factors. Let N be the smallest prime-minister number. Estimate N.

An estimate of $E > 0$ earns $\lfloor 20 \min\left(\frac{N}{E}, \frac{E}{N}\right) \rfloor$ points.

Proposed by: Yuan Yao

Answer: $\boxed{2^4 \cdot 3^3 \cdot 5^3 \cdot 7 = 378000}$

One heuristic for estimating the answer is that numbers of the form $p^q r^s$ for primes p, q, r, s with $p \neq r, q \neq s$ are primest. Thus, primest numbers are not very rare, so we can expect the answer to be relatively small with only a few distinct prime factors.

```
from operator import *

primes = []

primers = []

primests = []

for i in range(2,3000):
    prime_factors = 0
    primer_factors = 0
    temp = i
    for x in primes:
        if x > temp:
            break
        if temp % x == 0:
            prime_factors += 1
            while temp % x == 0:
                temp = temp // x
    if (prime_factors == 0):
        primes.append(i)
        continue
    elif (prime_factors in primes):
        primers.append(i)

    for x in primers:
        if i % x == 0:
            primer_factors += 1

    if (primer_factors in primers):
        primests.append(i)

def product(L):
    ans = 1
    for i in L:
        ans*= i
    return ans

def sum_prime_product(L, curr = ()):
    if (L == ()):
        #print(curr)
        if len(curr) in primes:
```

```
            return product(curr)
        return 0
    return sum_prime_product(L[1:], curr + (L[0],)) + sum_prime_product(L[1:], curr)

def count_primests(L, curr = ()):
    if (L == ()):
        if (sum_prime_product(curr) in primers):
            return 1
        return 0
    ans = 0
    for i in range(0,L[0]+1):
        ans += count_primests(L[1:], curr+(i,))
    return ans

def compute(L):
    ans = 1
    for i in range(len(L)):
        ans *= (primes[i]**L[i])
    return ans

def find_best(M, best = 2**20 * 3**5, curr = ()):
    num = compute(curr)
    if (num > best):
        return False
    if (count_primests(curr) in primests):
        print(num, curr)
        return num
    for i in range(1,M):
        result = find_best(M, best, curr + (i,))
        if (result == False):
            break
        elif (result < best):
            best = result
    return best


print("Answer:", find_best(30))
```

35. [**20**] Pascal has a triangle. In the nth row, there are $n+1$ numbers $a_{n,0}, a_{n,1}, a_{n,2}, \ldots, a_{n,n}$ where $a_{n,0} = a_{n,n} = 1$. For all $1 \le k \le n-1$, $a_{n,k} = a_{n-1,k} - a_{n-1,k-1}$. Let N be the value of the sum

$$\sum_{k=0}^{2018} \frac{|a_{2018,k}|}{\binom{2018}{k}}.$$

Estimate N.

An estimate of $E > 0$ earns $\lfloor 20 \cdot 2^{-|N-E|/70} \rfloor$ points.

Proposed by: Michael Ren

Answer: $\boxed{780.9280674537}$

A good estimate for this question is to use the fact that

$$\sum_{k=0}^{2018} |a_{2018,k}| = \frac{2^{2018}+2}{3},$$

the answer to Guts 17. This suggests that each $|a_{2018,k}|$ is roughly $\frac{1}{3}$ of its corresponding entry $\binom{2018}{k}$ in the usual Pascal's triangle, as the sum of the terms in the 2018th row of Pascal's triangle is 2^{2018}. This then gives an estimate of $\frac{2018}{3}$, which earns 6 points. Code for computing answer in Python 3:

```
import math
lists=[[1]]
for i in range(2018):
    newlist=[]
    for j in range(i):
        newlist.append(lists[-1][j+1]-lists[-1][j])
    lists.append([1]+newlist+[1])
big=math.factorial(2018)
sum=0
for i in range(2019):
    sum+=abs(lists[-1][i])/(big//math.factorial(i)//math.factorial(2018-i))
print(sum)
```

36. [**20**] An $n \times m$ maze is an $n \times m$ grid in which each cell is one of two things: a wall, or a blank. A maze is *solvable* if there exists a sequence of adjacent blank cells from the top left cell to the bottom right cell going through no walls. (In particular, the top left and bottom right cells must both be blank.) Let N be the number of solvable 5×5 mazes. Estimate N.

An estimate of $E > 0$ earns $\lfloor 20 \min\left(\frac{N}{E}, \frac{E}{N}\right)^2 \rfloor$ points.

Proposed by: John Michael Wu

Answer: $\boxed{1225194}$

The following code solves the problem in Python 3.

```
# dfs that returns all paths with no adjacent vertices other than those consecutive in the path
def dfs(graph,start,end,path):
    if start==end:
        return [path]
    paths=[]
    for child in graph[start]:
        skip=False
        if child in path:
            continue
        for vert in graph[child]:
            if vert in path[:-1]:
                skip=True
                break
        if not skip:
            paths=paths+dfs(graph,child,end,path[:]+[child])
    return paths
# construct graph representing 5x5 grid
graph={}
for a in range(5):
    for b in range(5):
        graph[(a,b)]=[]
for a in range(4):
    for b in range(5):
        graph[(a,b)].append((a+1,b));graph[(a+1,b)].append((a,b))
        graph[(b,a)].append((b,a+1));graph[(b,a+1)].append((b,a))
paths=dfs(graph,(0,0),(4,4),[(0,0)])
```

```
paths.sort(key=len)
intpaths=[0]*len(paths)
# convert paths to 25-bit binary integers
for i in range(len(paths)):
    for j in paths[i]:
        intpaths[i]+=2**(5*j[0]+j[1])
mazes=0
for j in range(2**23):
    k=2*j
    # disregard cases that are common and never valid
    if k&8912896==8912896 or k&34==34 or k&4472832==4472832 or k&1092==1092:
        continue
    for path in intpaths:
        # cehck if case has empty spaces along whole path
        if path&k==0:
            mazes+=1
            break
print(mazes)
```

Alternatively, the following code solves the problem in Java SE 8.

```
import java.util.*;

public class guts36 {

static int M = 5;
static int N = 5;

static long[] pow2 = new long[M * N];
static int[][] dir = new int[][] {new int[] {0, 1}, new int[] {1, 0}, new int[] {0, -1}, new int[]

public static void main(String[] args) {

pow2[0] = 1;
for (int i = 1; i < pow2.length; i++) {
pow2[i] = pow2[i - 1] * 2;
}

boolean[][] grid = new boolean[M][N];
grid[0][0] = true;
grid[M - 1][N - 1] = true;

int ans = 0;

for (long c = 0; c < pow2[M * N - 2]; c++) {

long d = c;
for (int b = 0; b < M * N - 2; b++) {
int i = (b + 1) / N;
int j = (b + 1) % N;
grid[i][j] = ((d & 1) > 0);
d >>= 1;
}
```

```
if (check(grid)) {
ans++;
}

}
System.out.println("answer: " + ans);

}

static int[] add(int[] a, int[] b) {
return new int[] {a[0] + b[0], a[1] + b[1]};
}

static boolean get(boolean[][] g, int[] a) {
return g[a[0]][a[1]];
}

static void set(boolean[][] g, int[] a, boolean v) {
g[a[0]][a[1]] = v;
}

static boolean valid(int[] a) {
return (a[0] >= 0) && (a[1] >= 0) && (a[0] < M) && (a[1] < N);
}

static boolean check(boolean[][] grid) {

Stack<int[]> q = new Stack<int[]>();
q.add(new int[] {0, 0});

boolean[][] reached = new boolean[M][N];
reached[0][0] = true;

while (!q.isEmpty()) {

int[] a = q.pop();
for (int[] d: dir) {
int[] b = add(a, d);
if (valid(b) && get(grid, b) && !get(reached, b)) {
if (b[0] == M - 1 && b[1] == N - 1) {
return true;
}
set(reached, b, true);
q.add(b);
}
}

}

return false;

}

}
```