HMMT November 2018

November 10, 2018

Team Round

1. [**15**] Four standard six-sided dice are rolled. Find the probability that, for each pair of dice, the product of the two numbers rolled on those dice is a multiple of 4.

   Proposed by: Michael Tang

   Answer: $\boxed{\frac{31}{432}}$

   If any two of the dice show an odd number, then this is impossible, so at most one of the dice can show an odd number. We take two cases:

   Case 1: If exactly one of the dice shows an odd number, then all three other dice must show a multiple of 4, which can only be the number 4. The probability that this occurs is $4 \cdot \frac{1}{2} \cdot \left(\frac{1}{6}\right)^3 = \frac{1}{108}$.

   Case 2: If all of the dice show even numbers, then the condition is satisfied. The probability that this occurs is $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$.

   The total probability is $\frac{1}{108} + \frac{1}{16} = \frac{31}{432}$.

2. [**20**] Alice starts with the number 0. She can apply 100 operations on her number. In each operation, she can either add 1 to her number, or square her number. After applying all operations, her score is the minimum distance from her number to any perfect square. What is the maximum score she can attain?

   Proposed by: Dhruv Rohatgi

   Answer: $\boxed{94}$

   Note that after applying the squaring operation, Alice's number will be a perfect square, so she can maximize her score by having a large number of adding operations at the end. However, her scores needs to be large enough that the many additions to not bring her close to a larger square. Hence the strategy is as follows: 2 additions to get to 2, 4 consecutive squares to get to 65536, and 94 more additions for a score of 94.

3. [**25**] For how many positive integers $n \leq 100$ is it true that $10n$ has exactly three times as many positive divisors as n has?

   Proposed by: James Lin

   Answer: $\boxed{28}$

   Let $n = 2^a 5^b c$, where $2, 5 \nmid c$. Then, the ratio of the number of divisors of $10n$ to the number of divisors of n is $\frac{a+2}{a+1}\frac{b+2}{b+1} = 3$. Solving for b, we find that $b = \frac{1-a}{2a+1}$. This forces $(a, b) = (0, 1), (1, 0)$. Therefore, the answers are of the form $2k$ and $5k$ whenever $\gcd(k, 10) = 1$. There are 50 positive numbers of the form $2k$ and 20 positive numbers of the form $5k$ less than or equal to 100. Of those 70 numbers, only $\frac{1}{2} \cdot \frac{4}{5}$ have k relatively prime to 10, so the answer is $70 \cdot \frac{1}{2} \cdot \frac{4}{5} = \boxed{28}$.

4. [**30**] Let a and b be real numbers greater than 1 such that $ab = 100$. The maximum possible value of $a^{(\log_{10} b)^2}$ can be written in the form 10^x for some real number x. Find x.

   Proposed by: James Lin

   Answer: $\boxed{\frac{32}{27}}$

   Let $p = \log_{10} a, q = \log_{10} b$. Since $a, b > 1$, p and q are positive. The condition $ab = 100$ translates to $p + q = 2$. We wish to maximize

   $$x = \log_{10} a^{(\log_{10} b)^2} = (\log_{10} a)(\log_{10} b)^2 = pq^2.$$

   By AM-GM,

   $$\frac{27}{4} pq^2 \leq \left(p + \frac{q}{2} + \frac{q}{2}\right)^3 = 8.$$

   Hence $pq^2 \leq \frac{32}{27}$ with equality when $p = \frac{2}{3}, q = \frac{4}{3}$.

5. [**35**] Find the sum of all positive integers n such that $1+2+\cdots+n$ divides

$$15[(n+1)^2+(n+2)^2+\cdots+(2n)^2].$$

Proposed by: Michael Ren

Answer: $\boxed{64}$

We can compute that $1+2+\cdots+n=\frac{n(n+1)}{2}$ and $(n+1)^2+(n+2)^2+\cdots+(2n)^2=\frac{2n(2n+1)(4n+1)}{6}-\frac{n(n+1)(2n+1)}{6}=\frac{n(2n+1)(7n+1)}{6}$, so we need $\frac{15(2n+1)(7n+1)}{3(n+1)}=\frac{5(2n+1)(7n+1)}{n+1}$ to be an integer. The remainder when $(2n+1)(7n+1)$ is divided by $(n+1)$ is 6, so after long division we need $\frac{30}{n+1}$ to be an integer. The solutions are one less than a divisor of 30 so the answer is

$$1+2+4+5+9+14+29=64.$$

6. [**45**] Triangle $\triangle PQR$, with $PQ=PR=5$ and $QR=6$, is inscribed in circle ω. Compute the radius of the circle with center on $\overline{QR}$ which is tangent to both ω and $\overline{PQ}$.

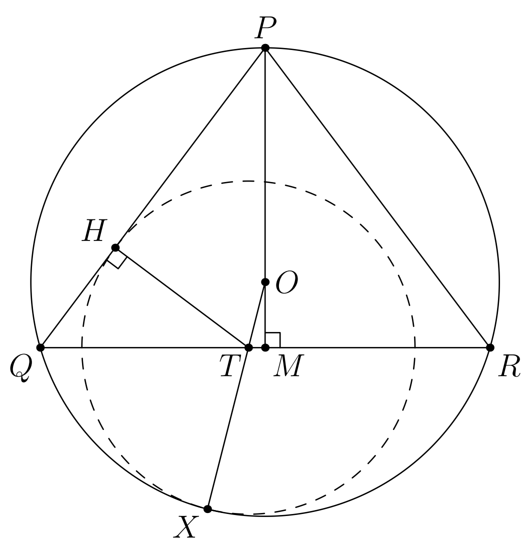
Proposed by: Michael Tang

Answer: $\boxed{\frac{20}{9}}$

Solution 1: Denote the second circle by γ. Let T and r be the center and radius of γ, respectively, and let X and H be the tangency points of γ with ω and $\overline{PQ}$, respectively. Let O be the center of ω, and let M be the midpoint of $\overline{QR}$. Note that $QM=MR=\frac{1}{2}QR=3$, so $\triangle PMQ$ and $\triangle PMR$ are $3-4-5$ triangles. Since $\triangle QHT\sim\triangle QMP$ and $HT=r$, we get $QT=\frac{5}{4}r$. Then $TM=QM-QT=3-\frac{5}{4}r$. By the extended law of sines, the circumradius of $\triangle PQR$ is $OP=\frac{PR}{2\sin\angle PQR}=\frac{5}{2(4/5)}=\frac{25}{8}$, so $OM=MP-OP=4-\frac{25}{8}=\frac{7}{8}$. Also, we have $OT=OX-XT=\frac{25}{8}-r$. Therefore, by the Pythagorean theorem,

$$\left(3-\tfrac{5}{4}r\right)^2+\left(\tfrac{7}{8}\right)^2=\left(\tfrac{25}{8}-r\right)^2.$$

This simplifies to $\frac{9}{16}r^2-\frac{5}{4}r=0$, so $r=\frac{5}{4}\cdot\frac{16}{9}=\boxed{\frac{20}{9}}$.



Solution 2: Following the notation of the previous solution, we compute the power of T with respect to ω in two ways. One of these is $-QT\cdot TR=-\frac{5}{4}r\left(6-\frac{5}{4}r\right)$. The other way is $-(OX-OT)(OX+OT)=-r(\frac{25}{4}-r)$. Equating these two yields $r=\frac{20}{9}$.

7. [**50**] A 5×5 grid of squares is filled with integers. Call a rectangle *corner-odd* if its sides are grid lines and the sum of the integers in its four corners is an odd number. What is the maximum possible number of corner-odd rectangles within the grid?

   Note: A rectangle must have four distinct corners to be considered *corner-odd*; i.e. no $1 \times k$ rectangle can be *corner-odd* for any positive integer k.

   Proposed by: Henrik Boecken

   Answer: $\boxed{60}$

   Consider any two rows and the five numbers obtained by adding the two numbers which share a given column. Suppose a of these are odd and b of these are even. The number of corner-odd rectangles with their sides contained in these two rows is ab. Since $a + b = 5$, we have $ab \le 6$. Therefore every pair of rows contains at most 6 corner-odd rectangles.

   There are $\binom{5}{2} = 10$ pairs of rows, so there are at most 60 corner-odd rectangles. Equality holds when we place 1 along one diagonal and 0 everywhere else.

8. [**55**] Tessa has a unit cube, on which each vertex is labeled by a distinct integer between 1 and 8 inclusive. She also has a deck of 8 cards, 4 of which are black and 4 of which are white. At each step she draws a card from the deck, and

   - if the card is black, she simultaneously replaces the number on each vertex by the sum of the three numbers on vertices that are distance 1 away from this vertex;
   - if the card is white, she simultaneously replaces the number on each vertex by the sum of the three numbers on vertices that are distance $\sqrt{2}$ away from this vertex.

   When Tessa finishes drawing all cards of the deck, what is the maximum possible value of a number that is on the cube?

   Proposed by: Yuan Yao

   Answer: $\boxed{42648}$

   The order of the deck does not matter as black cards and white cards commute, therefore we can assume that the cards are alternating black and white, and only worry about the arrangement of the numbers. After each pair of black and white cards, each number is replaced by the sum of two times the edge neighbors and three times the diagonally opposite number. We can compute that after four pairs of operations, the number at vertex V will be $1641v + 1640(d_1 + d_2 + d_3)$, where v is the number originally at v and d_1, d_2, d_3 are the numbers at diagonally adjacent vertices. Set $v = 8$ and $d_1, d_2, d_3 = 5, 6, 7$ in any order to obtain the maximum number 42648.

9. [**60**] Let A, B, C be points in that order along a line, such that $AB = 20$ and $BC = 18$. Let ω be a circle of nonzero radius centered at B, and let ℓ_1 and ℓ_2 be tangents to ω through A and C, respectively. Let K be the intersection of ℓ_1 and ℓ_2. Let X lie on segment $\overline{KA}$ and Y lie on segment $\overline{KC}$ such that $XY \parallel BC$ and XY is tangent to ω. What is the largest possible integer length for XY?

   Proposed by: James Lin

   Answer: $\boxed{35}$

   Note that B is the K-excenter of KXY, so XB is the angle bisector of $\angle AKY$. As AB and XY are parallel, $\angle XAB + 2\angle AXB = 180°$, so $\angle XBA = 180° - \angle AXB - \angle XAB$. This means that AXB is isosceles with $AX = AB = 20$. Similarly, $YC = BC = 18$. As KXY is similar to KAC, we have that $\frac{KX}{KY} = \frac{XA}{YC} = \frac{20}{18}$. Let $KA = 20x, KC = 18x$, so the Triangle Inequality applied to triangle KAC gives $KA < KC + AC \implies 20x < 18x + 38 \implies x < 19$. Then, $XY = AC \cdot \frac{KX}{KA} = 38 \cdot \frac{x-1}{x} = 38 - \frac{38}{x} < 36$, so the maximum possible integer length of XY is 35. The optimal configuration is achieved when the radius of ω becomes arbitrarily small and ℓ_1 and ℓ_2 are on opposite sides of AC.

10. [**65**] David and Evan are playing a game. Evan thinks of a positive integer N between 1 and 59, inclusive, and David tries to guess it. Each time David makes a guess, Evan will tell him whether the guess is greater than, equal to, or less than N. David wants to devise a strategy that will guarantee

that he knows N in five guesses. In David's strategy, each guess will be determined only by Evan's responses to any previous guesses (the first guess will always be the same), and David will only guess a number which satisfies each of Evan's responses. How many such strategies are there?

Note: David need not guess N within his five guesses; he just needs to know what N is after five guesses.

Proposed by: Anders Olsen

Answer: $\boxed{36440}$

We can represent each strategy as a binary tree labeled with the integers from 1 to 59, where David starts at the root and moves to the right child if he is too low and to the left child if he is too high. Our tree must have at most 6 layers as David must guess at most 5 times. Once David has been told that he guessed correctly or if the node he is at has no children, he will be sure of Evan's number. Consider the unique strategy for David when 59 is replaced with 63. This is a tree where every node in the first 5 layers has two children, and it can only be labeled in one way such that the strategy satisfies the given conditions. In order to get a valid strategy for 59, we only need to delete 4 of the vertices from this tree and relabel the vertices as necessary. Conversely, every valid strategy tree for 59 can be completed to the strategy tree for 63. If we delete a parent we must also delete its children. Thus, we can just count the number of ways to delete four nodes from the tree for 63 so that if a parent is deleted then so are its children. We cannot delete a node in the fourth layer, as that means we delete at least $1 + 2 + 4 = 7$ nodes. If we delete a node in the fifth layer, then we delete its two children as well, so in total we delete three nodes. There are now two cases: if we delete all four nodes from the sixth layer or if we delete one node in the fifth layer along with its children and another node in the sixth layer. There are $\binom{32}{4}$ ways to pick 4 from the sixth layer and $16 \cdot 30$ to pick one from the fifth layer along with its children and another node that is from the sixth layer, for a total of 36440.