# HMMT November 2018

**November 10, 2018**

## Theme Round

1. Square $CASH$ and regular pentagon $MONEY$ are both inscribed in a circle. Given that they do not share a vertex, how many intersections do these two polygons have?

   *Proposed by: Yuan Yao*

   **Answer:** $\boxed{8}$

   Pentagon $MONEY$ divides the circumference into 5 circular arcs, and each vertex of $CASH$ lies in a different arc. Then each side of $CASH$ will intersect two sides of $MONEY$, for a total of 8 intersections.

2. Consider the addition problem:

$$\begin{array}{ccccc} & C & A & S & H \\ + & & & M & E \\ \hline O & S & I & D & E \end{array}$$

   where each letter represents a base-ten digit, and $C, M, O \neq 0$. (Distinct letters are allowed to represent the same digit) How many ways are there to assign values to the letters so that the addition problem is true?

   *Proposed by: James Lin*

   **Answer:** $\boxed{0}$

   Clearly, $CASH$ and $ME$ cannot add up to 11000 or more, so $O = 1$ and $S = 0$. By examining the units digit, we find that $H = 0$. Then $CASH + ME < 9900 + 99 < 10000$, so there are no solutions.

3. $HOW$, $BOW$, and $DAH$ are equilateral triangles in a plane such that $WO = 7$ and $AH = 2$. Given that $D, A, B$ are collinear in that order, find the length of $BA$.

   *Proposed by: James Lin*

   **Answer:** $\boxed{11}$

   Note that $H \neq B$ since otherwise $DAB$ is an equilateral triangle. Let $M$ be the midpoint of $DA$, so $HB = 7\sqrt{3}$ and $HM = \sqrt{3}$, and $\angle HMB = 90^\circ$. By the Pythagorean theorem,

$$BM = \sqrt{(7\sqrt{3})^2 - (\sqrt{3})^2} = 12.$$

   Then $BA = BM - AM = 11$.

4. I have two cents and Bill has $n$ cents. Bill wants to buy some pencils, which come in two different packages. One package of pencils costs 6 cents for 7 pencils, and the other package of pencils costs a *dime for a dozen* pencils (i.e. 10 cents for 12 pencils). Bill notes that he can spend **all** $n$ of his cents on some combination of pencil packages to get $P$ pencils. However, if I *give my two cents* to Bill, he then notes that he can instead spend **all** $n + 2$ of his cents on some combination of pencil packages to get fewer than $P$ pencils. What is the smallest value of $n$ for which this is possible?

   Note: Both times Bill must spend **all** of his cents on pencil packages, i.e. have zero cents after either purchase.

   *Proposed by: James Lin*

   **Answer:** $\boxed{100}$

   Suppose that Bill buys $a$ packages of 7 and $b$ packages of 12 in the first scenario and $c$ packages of 7 and $d$ packages of 12 in the second scenario. Then we have the following system:

$$\begin{aligned} 6a + 10b &= n \\ 6c + 10d &= n + 2 \\ 7a + 12b &> 7c + 12d. \end{aligned}$$

Since the packages of 12 give more pencils per cent, we must have $b > d$. Subtract the first two equations and divide by 2 to get

$$3(c-a) - 5(b-d) = 1.$$

Note that the last inequality is $12(b-d) > 7(c-a)$. The minimal solutions to the equation with $b - d > 0$ are

$$(c-a, b-d) = (2,1), (7,4), (12,7), (17,10).$$

$(17, 10)$ is the first pair for which $12(b-d) > 7(c-a)$. Hence $b \geq 10$ so $n \geq 100$. We can easily verify that $(a, b, c, d, n) = (0, 10, 17, 0, 100)$ satisfies the system of equations.

5. Lil Wayne, the rain god, determines the weather. If Lil Wayne makes it rain on any given day, the probability that he makes it rain the next day is 75%. If Lil Wayne doesn't make it rain on one day, the probability that he makes it rain the next day is 25%. He decides not to make it rain today. Find the smallest positive integer $n$ such that the probability that Lil Wayne *makes it rain* $n$ days from today is greater than 49.9%.

   *Proposed by: Anders Olsen*

   **Answer:** $\boxed{9}$

   Let $p_n$ denote the probability that Lil Wayne makes it rain $n$ days from today. We have $p_0 = 0$ and

$$p_{n+1} = \frac{3}{4}p_n + \frac{1}{4}(1-p_n) = \frac{1}{4} + \frac{1}{2}p_n.$$

   This can be written as

$$p_{n+1} - \frac{1}{2} = \frac{1}{2}\left(p_n - \frac{1}{2}\right)$$

   and we can check that the solution of this recurrence is

$$p_n = \frac{1}{2} - \frac{1}{2^{n+1}}$$

   We want $\frac{1}{2^{n+1}} < \frac{1}{1000}$, which first occurs when $n = 9$.

6. Farmer James invents a new currency, such that for every positive integer $n \leq 6$, there exists an $n$-coin worth $n!$ cents. Furthermore, he has exactly $n$ copies of each $n$-coin. An integer $k$ is said to be *nice* if Farmer James can make $k$ cents using at least one copy of each type of coin. How many positive integers less than 2018 are nice?

   *Proposed by: Nikhil Reddy*

   **Answer:** $\boxed{210}$

   We use the factorial base, where we denote

$$(d_n \dots d_1)_* = d_n \times n! + \cdots + d_1 \times 1!$$

   The representation of $2018_{10}$ is $244002_*$ and the representation of $720_{10}$ is $100000_*$. The largest nice number less than $244002_*$ is $243321_*$. Notice that for the digit $d_i$ of a nice number, we can vary its value from 1 to $i$, while for a generic number in the factorial base, $d_{i-1}$ can vary from 0 to $i-1$. Hence we can map nice numbers to all numbers by truncating the last digit and reducing each previous digit by 1, and likewise reverse the procedure by increasing all digits by 1 and adding 1 at the end. Furthermore, this procedure preserves the ordering of numbers. Applying this procedure to $243321_*$ gives $13221_*$. We count from $0_*$ to $13221_*$ (since the first nice number is $1_*$), to get an answer of

$$13221_* + 1 = 210.$$

7. Ben "One Hunna Dolla" Franklin is flying a kite $KITE$ such that $IE$ is the perpendicular bisector of $KT$. Let $IE$ meet $KT$ at $R$. The midpoints of $KI, IT, TE, EK$ are $A, N, M, D$, respectively. Given that $[MAKE] = 18, IT = 10, [RAIN] = 4$, find $[DIME]$.

Note: $[X]$ denotes the area of the figure $X$.

*Proposed by: Michael Ren*

**Answer:** $\boxed{16}$

Let $[KIR] = [RIT] = a$ and $[KER] = [TER] = b$. We will relate all areas to $a$ and $b$. First,

$$[RAIN] = [RAI] + [INR] = \frac{1}{2}a + \frac{1}{2}a = a.$$

Next, we break up $[MAKE] = [MAD] + [AKD] + [DEM]$. We have

$$\begin{aligned}
[MAD] &= \frac{AD \cdot DM}{2} = \frac{1}{2} \cdot \frac{IE}{2} \cdot \frac{KT}{2} = \frac{[KITE]}{4} = \frac{a+b}{2} \\
[AKD] &= \frac{[KIE]}{4} = \frac{a+b}{4} \\
[DEM] &= \frac{[KTE]}{4} = \frac{b}{2}.
\end{aligned}$$

After adding these we get $[MAKE] = \frac{3a+5b}{4}$. We want to find

$$[DIME] = 2[IME] = [ITE] = a + b = \frac{4}{5}\left(\frac{3a+5b}{4}\right) + \frac{2}{5}a = \frac{4}{5} \cdot 18 + \frac{2}{5} \cdot 4 = 16.$$

8. Crisp All, a basketball player, is *dropping dimes* and nickels on a number line. Crisp drops a dime on every positive multiple of 10, and a nickel on every multiple of 5 that is not a multiple of 10. Crisp then starts at 0. Every second, he has a $\frac{2}{3}$ chance of jumping from his current location $x$ to $x+3$, and a $\frac{1}{3}$ chance of jumping from his current location $x$ to $x+7$. When Crisp jumps on either a dime or a nickel, he stops jumping. What is the probability that Crisp *stops on a dime*?

*Proposed by: James Lin*

**Answer:** $\boxed{\frac{20}{31}}$

Let "a 3" mean a move in which Crisp moves from $x$ to $x+3$, and "a 7" mean a move in which Crisp moves from $x$ to $x+7$. Note that Crisp stops precisely the first time his number of 3's and number of 7's differs by a multiple of 5, and that he'll *stop on a dime* if they differ by 0, and stop on a nickel if they differ by 5. This fact will be used without justification.

We split into two cases:

(a) Crisp begins with a 3. Rather than consider the integer Crisp is on, we'll count the difference, $n$, between his number of 3's and his number of 7's. Each 3 increases $n$ by 1, and each 7 decreases $n$ by 1. Currently, $n$ is 1. The probability of stopping on a dime, then, is the probability $n$ reaches 0 before $n$ reaches 5, where $n$ starts at 1. Let $a_i$ be the probability $n$ reaches 0 first, given a current position of $i$, for $i = 1, 2, 3, 4$. We desire $a_1$. We have the system of linear equations

$$\begin{aligned}
a_1 &= \frac{2}{3}a_2 + \frac{1}{3} \cdot 1 \\
a_2 &= \frac{2}{3}a_3 + \frac{1}{3}a_1 \\
a_3 &= \frac{2}{3}a_4 + \frac{1}{3}a_2 \\
a_4 &= \frac{2}{3} \cdot 0 + \frac{1}{3}a_3
\end{aligned}$$

From which we determine that $a_1 = \frac{15}{31}$.

(b) Crisp begins with a 7. Now, let $m$ be the difference between his number of 7's and his number of 3's. Let $b_i$ denote his probability of stopping on a dime, given his current position of $m = i$. We desire $b_1$. We have the system of linear equations

$$\begin{aligned} b_1 &= \frac{1}{3}b_2 + \frac{2}{3} \cdot 1 \\ b_2 &= \frac{1}{3}b_3 + \frac{2}{3}b_1 \\ b_3 &= \frac{1}{3}b_4 + \frac{2}{3}b_2 \\ b_4 &= \frac{1}{3} \cdot 0 + \frac{2}{3}b_3 \end{aligned}$$

From which we determine that $b_1 = \frac{30}{31}$.

We conclude that the answer is $\frac{2}{3}a_1 + \frac{1}{3}b_1 = \frac{2}{3} \cdot \frac{15}{31} + \frac{1}{3} \cdot \frac{30}{31} = \frac{20}{31}$.

9. Circle $\omega_1$ of radius 1 and circle $\omega_2$ of radius 2 are concentric. Godzilla inscribes square $CASH$ in $\omega_1$ and regular pentagon $MONEY$ in $\omega_2$. It then writes down all 20 (not necessarily distinct) distances between a vertex of $CASH$ and a vertex of $MONEY$ and multiplies them all together. What is the maximum possible value of his result?

*Proposed by: Michael Ren*

**Answer:** $\boxed{1048577 \text{ or } 2^{20}+1}$

We represent the vertices with complex numbers. Place the vertices of $CASH$ at $1, i, -1, -i$ and the vertices of $MONEY$ at $2\alpha, 2\alpha\omega, 2\alpha\omega^2, 2\alpha\omega^2, 2\alpha\omega^3, 2\alpha\omega^4$ with $|\alpha| = 1$ and $\omega = e^{\frac{2\pi i}{5}}$. We have that the product of distances from a point $z$ to the vertices of $CASH$ is $|(z-1)(z-i)(z+1)(z+i)| = |z^4 - 1|$, so we want to maximize $|(16\alpha^4 - 1)(16\alpha^4\omega^4 - 1)(16\alpha^4\omega^3 - 1)(16\alpha^4\omega^2 - 1)(16\alpha^4\omega - 1)|$, which just comes out to be $|2^{20}\alpha^{20} - 1|$. By the triangle inequality, this is at most $2^{20} + 1$, and it is clear that some $\alpha$ makes equality hold.

10. *One million bucks* (i.e. one million male deer) are in different cells of a $1000 \times 1000$ grid. The left and right edges of the grid are then glued together, and the top and bottom edges of the grid are glued together, so that the grid forms a doughnut-shaped torus. Furthermore, some of the bucks are *honest bucks*, who always tell the truth, and the remaining bucks are *dishonest bucks*, who never tell the truth. Each of the million bucks claims that "at most one of my neighboring bucks is an *honest buck*." A pair of neighboring bucks is said to be *buckaroo* if exactly one of them is an *honest buck*. What is the minimum possible number of *buckaroo* pairs in the grid?

Note: Two bucks are considered to be *neighboring* if their cells $(x_1, y_1)$ and $(x_2, y_2)$ satisfy either: $x_1 = x_2$ and $y_1 - y_2 \equiv \pm 1 \pmod{1000}$, or $x_1 - x_2 \equiv \pm 1 \pmod{1000}$ and $y_1 = y_2$.

*Proposed by: James Lin*

**Answer:** $\boxed{1200000}$

Note that each honest buck has at most one honest neighbor, and each dishonest buck has at least two honest neighbors. The connected components of honest bucks are singles and pairs. Then if there are $K$ honest bucks and $B$ buckaroo pairs, we get $B \geq 3K$. From the dishonest buck condition we get $B \geq 2(1000000 - K)$, so we conclude that $B \geq 1200000$. To find equality, partition the grid into five different parts with side $\sqrt{5}$, and put honest bucks on every cell in two of the parts.