HMMT February 2019

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Geometry

1. Let d be a real number such that every non-degenerate quadrilateral has at least two interior angles with measure less than d degrees. What is the minimum possible value for d?

2. In rectangle $ABCD$, points E and F lie on sides AB and CD respectively such that both AF and CE are perpendicular to diagonal BD. Given that BF and DE separate $ABCD$ into three polygons with equal area, and that $EF = 1$, find the length of BD.

3. Let AB be a line segment with length 2, and S be the set of points P on the plane such that there exists point X on segment AB with $AX = 2PX$. Find the area of S.

4. Convex hexagon $ABCDEF$ is drawn in the plane such that $ACDF$ and $ABDE$ are parallelograms with area 168. AC and BD intersect at G. Given that the area of AGB is 10 more than the area of CGB, find the smallest possible area of hexagon $ABCDEF$.

5. Isosceles triangle ABC with $AB = AC$ is inscribed in a unit circle Ω with center O. Point D is the reflection of C across AB. Given that $DO = \sqrt{3}$, find the area of triangle ABC.

6. Six unit disks $C_1, C_2, C_3, C_4, C_5, C_6$ are in the plane such that they don't intersect each other and C_i is tangent to C_{i+1} for $1 \leq i \leq 6$ (where $C_7 = C_1$). Let C be the smallest circle that contains all six disks. Let r be the smallest possible radius of C, and R the largest possible radius. Find $R - r$.

7. Let ABC be a triangle with $AB = 13, BC = 14, CA = 15$. Let H be the orthocenter of ABC. Find the radius of the circle with nonzero radius tangent to the circumcircles of AHB, BHC, CHA.

8. In triangle ABC with $AB < AC$, let H be the orthocenter and O be the circumcenter. Given that the midpoint of OH lies on BC, $BC = 1$, and the perimeter of ABC is 6, find the area of ABC.

9. In a rectangular box $ABCDEFGH$ with edge lengths $AB = AD = 6$ and $AE = 49$, a plane slices through point A and intersects edges BF, FG, GH, HD at points P, Q, R, S respectively. Given that $AP = AS$ and $PQ = QR = RS$, find the area of pentagon $APQRS$.

10. In triangle ABC, $AB = 13, BC = 14, CA = 15$. Squares ABB_1A_2, BCC_1B_2, CAA_1C_2 are constructed outside the triangle. Squares $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, $C_1C_2C_3C_4$ are constructed outside the hexagon $A_1A_2B_1B_2C_1C_2$. Squares $A_3B_4B_5A_6$, $B_3C_4C_5B_6$, $C_3A_4A_5C_6$ are constructed outside the hexagon $A_4A_3B_4B_3C_4C_3$. Find the area of the hexagon $A_5A_6B_5B_6C_5C_6$.