HMMO 2020

November 14–21, 2020

General Round

1. In the Cartesian plane, a line segment with midpoint $(2020, 11)$ has one endpoint at $(a, 0)$ and the other endpoint on the line $y = x$. Compute a.

2. Let T be a trapezoid with two right angles and side lengths 4, 4, 5, and $\sqrt{17}$. Two line segments are drawn, connecting the midpoints of opposite sides of T and dividing T into 4 regions. If the difference between the areas of the largest and smallest of these regions is d, compute $240d$.

3. Jody has 6 distinguishable balls and 6 distinguishable sticks, all of the same length. How many ways are there to use the sticks to connect the balls so that two disjoint non-interlocking triangles are formed? Consider rotations and reflections of the same arrangement to be indistinguishable.

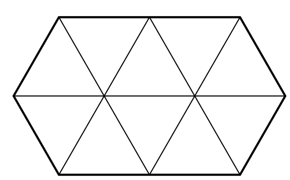
4. Nine fair coins are flipped independently and placed in the cells of a 3 by 3 square grid. Let p be the probability that no row has all its coins showing heads and no column has all its coins showing tails. If $p = \frac{a}{b}$ for relatively prime positive integers a and b, compute $100a + b$.

5. Compute the sum of all positive integers $a \leq 26$ for which there exist integers b and c such that $a + 23b + 15c - 2$ and $2a + 5b + 14c - 8$ are both multiples of 26.

6. A sphere is centered at a point with integer coordinates and passes through the three points $(2, 0, 0)$, $(0, 4, 0)$, $(0, 0, 6)$, but not the origin $(0, 0, 0)$. If r is the smallest possible radius of the sphere, compute r^2.

7. In triangle ABC with $AB = 8$ and $AC = 10$, the incenter I is reflected across side AB to point X and across side AC to point Y. Given that segment XY bisects AI, compute BC^2. (The incenter I is the center of the inscribed circle of triangle ABC.)

8. A bar of chocolate is made of 10 distinguishable triangles as shown below:



   How many ways are there to divide the bar, along the edges of the triangles, into two or more contiguous pieces?

9. In the Cartesian plane, a perfectly reflective semicircular room is bounded by the upper half of the unit circle centered at $(0, 0)$ and the line segment from $(-1, 0)$ to $(1, 0)$. David stands at the point $(-1, 0)$ and shines a flashlight into the room at an angle of $46°$ above the horizontal. How many times does the light beam reflect off the walls before coming back to David at $(-1, 0)$ for the first time?

10. A sequence of positive integers $a_1, a_2, a_3, \ldots$ satisfies

$$a_{n+1} = n \left\lfloor \frac{a_n}{n} \right\rfloor + 1$$

   for all positive integers n. If $a_{30} = 30$, how many possible values can a_1 take? (For a real number x, $\lfloor x \rfloor$ denotes the largest integer that is not greater than x.)