HMMO 2020

November 14–21, 2020

Theme Round

1. Chelsea goes to La Verde's at MIT and buys 100 coconuts, each weighing 4 pounds, and 100 honeydews, each weighing 5 pounds. She wants to distribute them among n bags, so that each bag contains at most 13 pounds of fruit. What is the minimum n for which this is possible?

2. In the future, MIT has attracted so many students that its buildings have become skyscrapers. Ben and Jerry decide to go ziplining together. Ben starts at the top of the Green Building, and ziplines to the bottom of the Stata Center. After waiting a seconds, Jerry starts at the top of the Stata Center, and ziplines to the bottom of the Green Building. The Green Building is 160 meters tall, the Stata Center is 90 meters tall, and the two buildings are 120 meters apart. Furthermore, both zipline at 10 meters per second. Given that Ben and Jerry meet at the point where the two ziplines cross, compute $100a$.

3. Harvard has recently built a new house for its students consisting of n levels, where the kth level from the top can be modeled as a 1-meter-tall cylinder with radius k meters. Given that the area of all the lateral surfaces (i.e. the surfaces of the external vertical walls) of the building is 35 percent of the total surface area of the building (including the bottom), compute n.

4. Points G and N are chosen on the interiors of sides ED and DO of unit square $DOME$, so that pentagon $GNOME$ has only two distinct side lengths. The sum of all possible areas of quadrilateral $NOME$ can be expressed as $\frac{a-b\sqrt{c}}{d}$, where a, b, c, d are positive integers such that $\gcd(a, b, d) = 1$ and c is square-free (i.e. no perfect square greater than 1 divides c). Compute $1000a + 100b + 10c + d$.

5. The classrooms at MIT are each identified with a positive integer (with no leading zeroes). One day, as President Reif walks down the Infinite Corridor, he notices that a digit zero on a room sign has fallen off. Let N be the original number of the room, and let M be the room number as shown on the sign. The smallest interval containing all possible values of $\frac{M}{N}$ can be expressed as $[\frac{a}{b}, \frac{c}{d})$ where a, b, c, d are positive integers with $\gcd(a, b) = \gcd(c, d) = 1$. Compute $1000a + 100b + 10c + d$.

6. The elevator buttons in Harvard's Science Center form a 3×2 grid of identical buttons, and each button lights up when pressed. One day, a student is in the elevator when all the other lights in the elevator malfunction, so that only the buttons which are lit can be seen, but one cannot see which floors they correspond to. Given that at least one of the buttons is lit, how many distinct arrangements can the student observe? (For example, if only one button is lit, then the student will observe the same arrangement regardless of which button it is.)

7. While waiting for their food at a restaurant in Harvard Square, Ana and Banana draw 3 squares $\square_1, \square_2, \square_3$ on one of their napkins. Starting with Ana, they take turns filling in the squares with integers from the set $\{1, 2, 3, 4, 5\}$ such that no integer is used more than once. Ana's goal is to minimize the minimum value M that the polynomial $a_1x^2 + a_2x + a_3$ attains over all real x, where a_1, a_2, a_3 are the integers written in $\square_1, \square_2, \square_3$ respectively. Banana aims to maximize M. Assuming both play optimally, compute the final value of $100a_1 + 10a_2 + a_3$.

8. After viewing the John Harvard statue, a group of tourists decides to estimate the distances of nearby locations on a map by drawing a circle, centered at the statue, of radius $\sqrt{n}$ inches for each integer $2020 \le n \le 10000$, so that they draw 7981 circles altogether. Given that, on the map, the Johnston Gate is 10-inch line segment which is entirely contained between the smallest and the largest circles, what is the minimum number of points on this line segment which lie on one of the drawn circles? (The endpoint of a segment is considered to be on the segment.)

9. While waiting for their next class on Killian Court, Alesha and Belinda both write the same sequence S on a piece of paper, where S is a 2020-term strictly increasing geometric sequence with an integer common ratio r. Every second, Alesha erases the two smallest terms on her paper and replaces them with their geometric mean, while Belinda erases the two largest terms in her paper and replaces them with their geometric mean. They continue this process until Alesha is left with a single value A and Belinda is left with a single value B. Let r_0 be the minimal value of r such that $\frac{A}{B}$ is an integer. If d is the number of positive factors of r_0, what is the closest integer to $\log_2 d$?

10. Sean enters a classroom in the Memorial Hall and sees a 1 followed by 2020 0's on the blackboard. As he is early for class, he decides to go through the digits from right to left and independently erase the nth digit from the left with probability $\frac{n-1}{n}$. (In particular, the 1 is never erased.) Compute the expected value of the number formed from the remaining digits when viewed as a base-3 number. (For example, if the remaining number on the board is 1000, then its value is 27.)