HMMT Spring 2021

March 06, 2021

Algebra and Number Theory Round

1. Compute the sum of all positive integers n for which the expression

$$\frac{n+7}{\sqrt{n-1}}$$

is an integer.

2. Compute the number of ordered pairs of integers (a, b), with $2 \leq a, b \leq 2021$, that satisfy the equation

$$a^{\log_b\left(a^{-4}\right)} = b^{\log_a\left(ba^{-3}\right)}.$$

3. Among all polynomials $P(x)$ with integer coefficients for which $P(-10) = 145$ and $P(9) = 164$, compute the smallest possible value of $|P(0)|$.

4. Suppose that $P(x, y, z)$ is a homogeneous degree 4 polynomial in three variables such that $P(a, b, c) = P(b, c, a)$ and $P(a, a, b) = 0$ for all real a, b, and c. If $P(1, 2, 3) = 1$, compute $P(2, 4, 8)$.

   Note: $P(x, y, z)$ is a homogeneous degree 4 polynomial if it satisfies $P(ka, kb, kc) = k^4 P(a, b, c)$ for all real k, a, b, c.

5. Let n be the product of the first 10 primes, and let

$$S = \sum_{xy|n} \varphi(x) \cdot y,$$

where $\varphi(x)$ denotes the number of positive integers less than or equal to x that are relatively prime to x, and the sum is taken over ordered pairs (x, y) of positive integers for which xy divides n. Compute $\frac{S}{n}$.

6. Suppose that m and n are positive integers with $m < n$ such that the interval $[m, n)$ contains more multiples of 2021 than multiples of 2000. Compute the maximum possible value of $n - m$.

7. Suppose that x, y, and z are complex numbers of equal magnitude that satisfy

$$x + y + z = -\frac{\sqrt{3}}{2} - i\sqrt{5}$$

and

$$xyz = \sqrt{3} + i\sqrt{5}.$$

If $x = x_1 + ix_2$, $y = y_1 + iy_2$, and $z = z_1 + iz_2$ for real x_1, x_2, y_1, y_2, z_1, and z_2, then

$$(x_1x_2 + y_1y_2 + z_1z_2)^2$$

can be written as $\frac{a}{b}$ for relatively prime positive integers a and b. Compute $100a + b$.

8. For positive integers a and b, let $M(a, b) = \frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}$, and for each positive integer $n \geq 2$, define

$$x_n = M(1, M(2, M(3, \ldots, M(n-2, M(n-1, n)) \ldots))).$$

Compute the number of positive integers n such that $2 \leq n \leq 2021$ and $5x_n^2 + 5x_{n+1}^2 = 26x_nx_{n+1}$.

9. Let f be a monic cubic polynomial satisfying $f(x) + f(-x) = 0$ for all real numbers x. For all real numbers y, define $g(y)$ to be the number of distinct real solutions x to the equation $f(f(x)) = y$. Suppose that the set of possible values of $g(y)$ over all real numbers y is exactly $\{1, 5, 9\}$. Compute the sum of all possible values of $f(10)$.

10. Let S be a set of positive integers satisfying the following two conditions:

- For each positive integer n, at least one of $n, 2n, \ldots, 100n$ is in S.
- If a_1, a_2, b_1, b_2 are positive integers such that $\gcd(a_1a_2, b_1b_2) = 1$ and $a_1b_1, a_2b_2 \in S$, then $a_2b_1, a_1b_2 \in S$.

Suppose that S has natural density r. Compute the minimum possible value of $\lfloor 10^5 r \rfloor$.

Note: S has natural density r if $\frac{1}{n}|S \cap \{1, ..., n\}|$ approaches r as n approaches ∞.