HMMT Spring 2021

March 06, 2021

Algebra and Number Theory Round

1. Compute the sum of all positive integers n for which the expression

$$\frac{n+7}{\sqrt{n-1}}$$

is an integer.

Proposed by: Ryan Kim

Answer: $\boxed{89}$

Solution: We know $\sqrt{n-1}$ must be a positive integer, because the numerator is a positive integer, and the square root of an integer cannot be a non-integer rational. From this,

$$\frac{n+7}{\sqrt{n-1}} = \sqrt{n-1} + \frac{8}{\sqrt{n-1}}$$

is a positive integer, so we $\sqrt{n-1}$ must be a positive integer that divides 8. There are 4 such positive integers: $1, 2, 4, 8$, which give $n = 2, 5, 17, 65$, so the answer is 89.

2. Compute the number of ordered pairs of integers (a, b), with $2 \le a, b \le 2021$, that satisfy the equation

$$a^{\log_b\left(a^{-4}\right)} = b^{\log_a\left(ba^{-3}\right)}.$$

Proposed by: Vincent Bian

Answer: $\boxed{43}$

Solution: Taking $\log_a$ of both sides and simplifying tives

$$-4\log_b a = (\log_a b)^2 - 3\log_a b.$$

Plugging in $x = \log_a b$ and using $\log_b a = \frac{1}{\log_a b}$ gives

$$x^3 - 3x^2 + 4 = 0.$$

We can factor the polynomial as $(x-2)(x-2)(x+1)$, meaning $b = a^2$ or $b = a^{-1}$. The second case is impossible since both a and b are positive integers. So, we need only count the number of $1 < a, b \le 2021$ for which $b = a^2$, which is $\lfloor\sqrt{2021}\rfloor - 1 = 43$.

3. Among all polynomials $P(x)$ with integer coefficients for which $P(-10) = 145$ and $P(9) = 164$, compute the smallest possible value of $|P(0)|$.

Proposed by: Carl Schildkraut

Answer: $\boxed{25}$

Solution: Since $a - b|P(a) - P(b)$ for any integer polynomial P and integers a and b, we require that $10|P(0) - P(-10)$ and $9|P(0) - P(9)$. So, we are looking for an integer a near 0 for which

$$a \equiv 5 \bmod 10, \; a \equiv 2 \bmod 9.$$

The smallest such positive integer is 65, and the smallest such negative integer is -25. This is achievable, for example, if $P(x) = 2x^2 + 3x - 25$, so our answer is 25.

4. Suppose that $P(x, y, z)$ is a homogeneous degree 4 polynomial in three variables such that $P(a, b, c) = P(b, c, a)$ and $P(a, a, b) = 0$ for all real a, b, and c. If $P(1, 2, 3) = 1$, compute $P(2, 4, 8)$.

   Note: $P(x, y, z)$ is a homogeneous degree 4 polynomial if it satisfies $P(ka, kb, kc) = k^4 P(a, b, c)$ for all real k, a, b, c.

   Proposed by: Milan Haiman

   Answer: $\boxed{56}$

   Solution: Since $P(a, a, b) = 0$, $(x - y)$ is a factor of P, which means $(y - z)$ and $(z - x)$ are also factors by the symmetry of the polynomial. So,

$$\frac{P(x, y, z)}{(x-y)(y-z)(z-x)}$$

   is a symmetric homogeneous degree 1 polynomial, so it must be $k(x + y + z)$ for some real k. So, the answer is

$$\frac{P(2, 4, 8)}{P(1, 2, 3)} = \frac{(2+4+8)(2-4)(4-8)(8-2)}{(1+2+3)(1-2)(2-3)(3-1)} = 56.$$

5. Let n be the product of the first 10 primes, and let

$$S = \sum_{xy|n} \varphi(x) \cdot y,$$

   where $\varphi(x)$ denotes the number of positive integers less than or equal to x that are relatively prime to x, and the sum is taken over ordered pairs (x, y) of positive integers for which xy divides n. Compute $\frac{S}{n}$.

   Proposed by: Hahn Lheem

   Answer: $\boxed{1024}$

   Solution 1: We see that, for any positive integer n,

$$S = \sum_{xy|n} \varphi(x) \cdot y = \sum_{x|n} \varphi(x) \left(\sum_{y|\frac{n}{x}} y \right) = \sum_{x|n} \varphi(x) \sigma\left(\frac{n}{x}\right).$$

   Since φ and σ are both weakly multiplicative (if x and y are relatively prime, then $\varphi(xy) = \varphi(x)\varphi(y)$ and $\sigma(xy) = \sigma(x)\sigma(y)$), we may break this up as

$$\prod_p (\varphi(p) + \sigma(p)),$$

   where the product is over all primes that divide n. This is simply $2^{10}n$, giving an answer of $2^{10} = 1024$.

   Solution 2: We recall that

$$\sum_{d|n} \varphi(d) = n.$$

   So, we may break up the sum as

$$S = \sum_{xy|n} \varphi(x) \cdot y = \sum_{y|n} y \sum_{x|\frac{n}{y}} \varphi(x) = \sum_{y|n} y \left(\frac{n}{y}\right),$$

   so S is simply n times the number of divisors of n. This number is $2^{10} = 1024$.

Solution 3: When constructing a term in the sum, for each prime p dividing n, we can choose to include p in x, or in y, or in neither. This gives a factor of $p-1$, p, or 1, respectively. Thus we can factor the sum as

$$S = \prod_{p|n}(p - 1 + p + 1) = \prod_{p|n} 2p = 2^{10}n.$$

So the answer is 1024.

6. Suppose that m and n are positive integers with $m < n$ such that the interval $[m, n)$ contains more multiples of 2021 than multiples of 2000. Compute the maximum possible value of $n - m$.

Proposed by: Carl Schildkraut

Answer: $\boxed{191999}$

Solution: Let $a = 2021$ and $b = 2000$. It is clear that we may increase $y - x$ unless both $x - 1$ and $y + 1$ are multiples of b, so we may assume that our interval is of length $b(k + 1) - 1$, where there are k multiples of b in our interval. There are at least $k + 1$ multiples of a, and so it is of length at least $ak + 1$. We thus have that

$$ak + 1 \le b(k + 1) - 1 \implies (a - b)k \le b - 2 \implies k \le \left\lfloor \frac{b-2}{a-b} \right\rfloor.$$

So, the highest possible value of k is 95, and this is achievable by the Chinese remainder theorem, giving us an answer of 191999.

7. Suppose that x, y, and z are complex numbers of equal magnitude that satisfy

$$x + y + z = -\frac{\sqrt{3}}{2} - i\sqrt{5}$$

and

$$xyz = \sqrt{3} + i\sqrt{5}.$$

If $x = x_1 + ix_2$, $y = y_1 + iy_2$, and $z = z_1 + iz_2$ for real x_1, x_2, y_1, y_2, z_1, and z_2, then

$$(x_1x_2 + y_1y_2 + z_1z_2)^2$$

can be written as $\frac{a}{b}$ for relatively prime positive integers a and b. Compute $100a + b$.

Proposed by: Akash Das

Answer: $\boxed{1516}$

Solution: From the conditions, it is clear that a, b, c all have magnitude $\sqrt{2}$. Conjugating the first equation gives $2(\frac{ab+bc+ca}{abc}) = -\frac{\sqrt{3}}{2} + i\sqrt{5}$, which means $ab + bc + ca = (-\frac{\sqrt{3}}{4} + i\frac{\sqrt{5}}{2})(\sqrt{3} + i\sqrt{5}) = \frac{-13+i\sqrt{15}}{4}$. Then,

$$\begin{aligned} a_1a_2 + b_1b_2 + c_1c_2 &= \frac{1}{2}\operatorname{Im}(a^2 + b^2 + c^2) \\ &= \frac{1}{2}\operatorname{Im}((a + b + c)^2) - \operatorname{Im}(ab + bc + ca) \\ &= \frac{\sqrt{15}}{4}, \end{aligned}$$

so the answer is 1516.

Remark:

$$\{a, b, c\} = \left\{\frac{-\sqrt{3} - i\sqrt{5}}{2}, \frac{-3\sqrt{3} - i\sqrt{5}}{4}, \frac{3\sqrt{3} - i\sqrt{5}}{4}\right\}$$

8. For positive integers a and b, let $M(a, b) = \frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}$, and for each positive integer $n \geq 2$, define

$$x_n = M(1, M(2, M(3, \ldots, M(n-2, M(n-1, n)) \ldots))).$$

Compute the number of positive integers n such that $2 \leq n \leq 2021$ and $5x_n^2 + 5x_{n+1}^2 = 26x_nx_{n+1}$.

Proposed by: Hahn Lheem

Answer: $\boxed{20}$

Solution: The desired condition is that $x_n = 5x_{n+1}$ or $x_{n+1} = 5x_n$.

Note that for any prime p, we have $\nu_p(M(a, b)) = |\nu_p(a) - \nu_p(b)|$. Furthermore, $\nu_p(M(a, b)) \equiv \nu_p(a) + \nu_p(b) \bmod 2$. So, we have that

$$\nu_p(x_n) \equiv \nu_p(1) + \nu_p(2) + \cdots + \nu_p(n) \bmod 2.$$

Subtracting gives that $\nu_p(x_{n+1}) - \nu_p(x_n) \equiv \nu_p(n+1) \bmod 2$. In particular, for $p \neq 5$, $\nu_p(n+1)$ must be even, and $\nu_5(n+1)$ must be odd. So $n+1$ must be a 5 times a perfect square. There are $\left\lfloor \sqrt{\frac{2021}{5}} \right\rfloor = 20$ such values of n in the interval $[2, 2021]$.

Now we show that it is sufficient for $n + 1$ to be 5 times a perfect square. The main claim is that if $B > 0$ and a sequence $a_1, a_2, \ldots, a_B$ of nonnegative real numbers satisfies $a_n \leq B + \sum_{i<n} a_i$ for all $1 \leq n \leq N$, then

$$\Bigg| a_1 - \Bigg| a_2 - \Bigg| \cdots - \big| a_{N-1} - a_N \big| \cdots \Bigg| \Bigg| \Bigg| \leq B.$$

This can be proved by a straightforward induction on N. We then apply this claim, with $B = 1$, to the sequence $a_i = \nu_p(i)$; it is easy to verify that this sequence satisfies the condition. This gives

$$\nu_p(x_n) = \Bigg| \nu_p(1) - \Bigg| \nu_p(2) - \Bigg| \cdots - \big| \nu_p(n-1) - \nu_p(n) \big| \cdots \Bigg| \Bigg| \Bigg| \leq 1,$$

so $\nu_p(x_n)$ must be equal to $\big(\nu_p(1) + \cdots + \nu_p(n)\big) \bmod 2$. Now suppose $n + 1 = 5k^2$ for some k; then $\nu_p(n + 1) \equiv 0 \bmod 2$ for $p \neq 5$ and $\nu_5(n + 1) \equiv 1 \bmod 2$. Therefore $\nu_p(x_{n+1}) = \nu_p(x_n)$ for $p \neq 5$, and $\nu_5(x_{n+1}) = (\nu_5(x_n) + 1) \bmod 2$, and this implies $x_{n+1}/x_n \in \{1/5, 5\}$ as we wanted.

9. Let f be a monic cubic polynomial satisfying $f(x) + f(-x) = 0$ for all real numbers x. For all real numbers y, define $g(y)$ to be the number of distinct real solutions x to the equation $f(f(x)) = y$. Suppose that the set of possible values of $g(y)$ over all real numbers y is exactly $\{1, 5, 9\}$. Compute the sum of all possible values of $f(10)$.

Proposed by: Sujay Kazi

Answer: $\boxed{970}$

Solution: We claim that we must have $f(x) = x^3 - 3x$. First, note that the condition $f(x) + f(-x) = 0$ implies that f is odd. Combined with f being monic, we know that $f(x) = x^3 + ax$ for some real number a. Note that a must be negative; otherwise $f(x)$ and $f(f(x))$ would both be increasing and 1 would be the only possible value of $g(y)$.

Now, consider the condition that the set of possible values of $g(y)$ is $\{1, 5, 9\}$. The fact that we can have $g(y) = 9$ means that some horizontal line crosses the graph of $f(f(x))$ 9 times. Since $f(f(x))$ has degree 9, this means that its graph will have 4 local maxima and 4 local minima.

Now, suppose we start at some value of y such that $g(y) = 9$, and slowly increase y. At some point, the value of $g(y)$ will decrease. This happens when y is equal to a local maximum of f. Since $g(y)$

must jump from 9 down to 5, all four local maxima must have the same value. Similarly, all four local minima must also have the same value. Since f is odd, it suffices to just consider the four local maxima.

The local maximum of $f(x)$ occurs when $3x^2+a=0$. For convenience, let $a=-3b^2$, so $f(x)=x^3-3b^2x$. Then, the local maximum is at $x=-b$, and has a value of $f(-b)=2b^3$.

We consider the local maxima of $f(f(x))$ next. They occur either when $x=-b$ (meaning $f(x)$ is at a local maximum) or $f(x)=-b$. If $f(x)=-b$, then $f(f(x))=f(-b)=2b^3$. Thus, we must have $f(f(-b))=f(2b^3)=2b^3$.

This yields the equation

$$f(2b^3)=8b^9-3b^2\cdot 2b^3=2b^3$$

which factors as $2b^3\left(b^2-1\right)\left(2b^2+1\right)^2$. The only possible value of b^2 is 1. Thus, $f(x)=x^3-3x$, and our answer is $10^3-3\cdot 10=970$.

10. Let S be a set of positive integers satisfying the following two conditions:

- For each positive integer n, at least one of $n, 2n, \ldots, 100n$ is in S.
- If a_1, a_2, b_1, b_2 are positive integers such that $\gcd(a_1a_2, b_1b_2)=1$ and $a_1b_1, a_2b_2\in S$, then $a_2b_1, a_1b_2\in S$.

Suppose that S has natural density r. Compute the minimum possible value of $\lfloor 10^5 r\rfloor$.

Note: S has natural density r if $\frac{1}{n}|S\cap\{1,...,n\}|$ approaches r as n approaches ∞.

Proposed by: Milan Haiman

Answer: $\boxed{396}$

Solution: The optimal value of r is $\frac{1}{252}$. This is attained by letting S be the set of integers n for which $\nu_2(n)\equiv 4 \bmod 5$ and $\nu_3(n)\equiv 1 \bmod 2$.

Let S be a set of positive integers satisfying the two conditions. For each prime p, let $A_p=\{\nu_p(n): n\in S\}$. We claim that in fact S is precisely the set of positive integers n for which $\nu_p(n)\in A_p$ for each prime p.

Let p be prime and suppose that $a_1p^{e_1}, a_2p^{e_2}\in S$, with $p\nmid a_1, a_2$. Then, setting $b_1=p^{e_1}$ and $b_2=p^{e_2}$ in the second condition gives that $a_1p^{e_2}\in S$ as well. So, if we have an integer n for which $\nu_p(n)\in A_p$ for each prime p, we can start with any element n' of S and apply this step for each prime divisor of n and n' to obtain $n\in S$.

Now we deal with the first condition. Let n be any positive integer. We will compute the least positive integer m such that $mn\in S$. By the above result, we can work with each prime separately. For a given prime p, let e_p be the least element of A_p with $e_p\geq\nu_p(n)$. Then we must have $\nu_p(m)\geq e_p-\nu_p(n)$, and equality for all primes p is sufficient. So, if the elements of A_p are $c_{p,1}<c_{p,2}<c_{p,3}<c_{p,4}<\ldots$, then

$$c_p=\max(c_{p,1}, c_{p,2}-c_{p,1}-1, c_{p,3}-c_{p,2}-1, c_{p,4}-c_{p,3}-1,\ldots)$$

is the worst case value for $\nu_p(m)$.

We conclude two things from this. First, we must have $\prod_p p^{c_p}\leq 100$ by condition 1, and in fact this is sufficient. Second, since we only care about c_p and would like to minimize r, the optimal choice for A_p is an arithmetic progression with first term c_p and common difference c_p+1. So we assume that each A_p is of this form.

Let $t=\prod_p p^{c_p}$. We now compute r. Note that S is the set of integers n such that for each prime p,

$$n\equiv ap^{k(c_p+1)-1} \bmod p^{k(c_p+1)}$$

for some positive integers a, k with $a < p$. This means that each prime p contributes a factor of

$$\frac{p-1}{p^{c_p+1}} + \frac{p-1}{p^{2c_p+2}} + \frac{p-1}{p^{3c_p+3}} + \cdots = \frac{p-1}{p^{c_p+1}-1} = \frac{1}{1+p+\cdots+p^{c_p}}$$

to the density of S. Multiplying over all primes p gives $r = \frac{1}{\sigma(t)}$, where $\sigma(t)$ is the sum of divisors of t. So, it suffices to maximize $\sigma(t)$ for $t \leq 100$. By inspection, $t = 96$ is optimal, giving $r = \frac{1}{252}$.