HMMT Spring 2021

March 06, 2021

Combinatorics Round

1. Leo the fox has a 5 by 5 checkerboard grid with alternating red and black squares. He fills in the grid with the numbers $1, 2, 3, \ldots, 25$ such that any two consecutive numbers are in adjacent squares (sharing a side) and each number is used exactly once. He then computes the sum of the numbers in the 13 squares that are the same color as the center square. Compute the maximum possible sum Leo can obtain.

2. Ava and Tiffany participate in a knockout tournament consisting of a total of 32 players. In each of 5 rounds, the remaining players are paired uniformly at random. In each pair, both players are equally likely to win, and the loser is knocked out of the tournament. The probability that Ava and Tiffany play each other during the tournament is $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

3. Let N be a positive integer. Brothers Michael and Kylo each select a positive integer less than or equal to N, independently and uniformly at random. Let p_N denote the probability that the product of these two integers has a units digit of 0. The maximum possible value of p_N over all possible choices of N can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

4. Let $S = \{1, 2, \ldots, 9\}$. Compute the number of functions $f\colon S \to S$ such that, for all $s \in S$, $f(f(f(s))) = s$ and $f(s) - s$ is not divisible by 3.

5. Teresa the bunny has a fair 8-sided die. Seven of its sides have fixed labels $1, 2, \ldots, 7$, and the label on the eighth side can be changed and begins as 1. She rolls it several times, until each of $1, 2, \ldots, 7$ appears at least once. After each roll, if k is the smallest positive integer that she has not rolled so far, she relabels the eighth side with k. The probability that 7 is the last number she rolls is $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

6. A light pulse starts at a corner of a reflective square. It bounces around inside the square, reflecting off of the square's perimeter n times before ending in a different corner. The path of the light pulse, when traced, divides the square into exactly 2021 regions. Compute the smallest possible value of n.

7. Let $S = \{1, 2, \ldots, 2021\}$, and let $\mathcal{F}$ denote the set of functions $f : S \to S$. For a function $f \in \mathcal{F}$, let

$$T_f = \left\{ f^{2021}(s) \;:\; s \in S \right\},$$

where $f^{2021}(s)$ denotes $f(f(\cdots(f(s))\cdots))$ with 2021 copies of f. Compute the remainder when

$$\sum_{f \in \mathcal{F}} |T_f|$$

is divided by the prime 2017, where the sum is over all functions f in $\mathcal{F}$.

8. Compute the number of ways to fill each cell in a 8×8 square grid with one of the letters $H, M,$ or T such that every 2×2 square in the grid contains the letters H, M, M, T in some order.

9. An up-right path between two lattice points P and Q is a path from P to Q that takes steps of length 1 unit either up or to the right.

   How many up-right paths from $(0, 0)$ to $(7, 7)$, when drawn in the plane with the line $y = x - 2.021$, enclose exactly one bounded region below that line?

10. Jude repeatedly flips a coin. If he has already flipped n heads, the coin lands heads with probability $\frac{1}{n+2}$ and tails with probability $\frac{n+1}{n+2}$. If Jude continues flipping forever, let p be the probability that he flips 3 heads in a row at some point. Compute $\lfloor 180p \rfloor$.