HMMT Spring 2021

March 06, 2021

Geometry Round

1. A circle contains the points $(0, 11)$ and $(0, -11)$ on its circumference and contains all points (x, y) with $x^2 + y^2 < 1$ in its interior. Compute the largest possible radius of the circle.

   Proposed by: Carl Schildkraut

   Answer: $\boxed{61}$

   Solution: Such a circle will be centered at $(t, 0)$ for some t; without loss of generality, let $t > 0$. Our conditions are that

   $$t^2 + 11^2 = r^2$$

   and

   $$r \geq t + 1.$$

   So, $t^2 \leq (r-1)^2$, which means

   $$(r-1)^2 + 11^2 \geq r^2 \implies 122 \geq 2r,$$

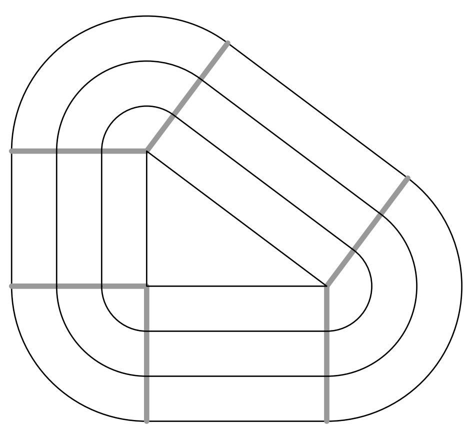
   so our answer is 61 ($r = 61$ is attainable with $t = 60$).

2. Let X_0 be the interior of a triangle with side lengths 3, 4, and 5. For all positive integers n, define X_n to be the set of points within 1 unit of some point in X_{n-1}. The area of the region outside X_{20} but inside X_{21} can be written as $a\pi + b$, for integers a and b. Compute $100a + b$.

   Proposed by: Hahn Lheem

   Answer: $\boxed{4112}$

   Solution:



   X_n is the set of points within n units of some point in X_0. The diagram above shows X_0, X_1, X_2, and X_3. As seen above it can be verified that X_n is the union of

   - X_0,
   - three rectangles of height n with the sides of X_0 as bases, and
   - three sectors of radius n centered at the vertices and joining the rectangles

   Therefore the total area of X_n is

   $$[X_0] + n \cdot \text{perimeter}(X_0) + n^2\pi.$$

Since X_{n-1} is contained entirely within X_n, the area within X_n but not within X_{n-1} is

$$\text{perimeter}(X_0) + (2n-1)\pi.$$

Since X_0 is a $(3,4,5)$ triangle, and $n = 21$, this is $12 + 41\pi$.

3. Triangle ABC has a right angle at C, and D is the foot of the altitude from C to AB. Points L, M, and N are the midpoints of segments AD, DC, and CA, respectively. If $CL = 7$ and $BM = 12$, compute BN^2.

   Proposed by: Hahn Lheem

   Answer: $\boxed{193}$

   Solution: Note that CL, BM, and BN are corresponding segments in the similar triangles $\triangle ACD \sim \triangle CBD \sim \triangle ABC$. So, we have

   $$CL : BM : BN = AD : CD : AC.$$

   Since $AD^2 + CD^2 = AC^2$, we also have $CL^2 + BM^2 = BN^2$, giving an answer of $49 + 144 = 193$.

4. Let $ABCD$ be a trapezoid with $AB \parallel CD$, $AB = 5$, $BC = 9$, $CD = 10$, and $DA = 7$. Lines BC and DA intersect at point E. Let M be the midpoint of CD, and let N be the intersection of the circumcircles of $\triangle BMC$ and $\triangle DMA$ (other than M). If $EN^2 = \frac{a}{b}$ for relatively prime positive integers a and b, compute $100a + b$.

   Proposed by: Milan Haiman

   Answer: $\boxed{90011}$

   Solution: From $\triangle EAB \sim \triangle EDC$ with length ratio $1:2$, we have $EA = 7$ and $EB = 9$. This means that A, B, M are the midpoints of the sides of $\triangle ECD$. Let N' be the circumcenter of $\triangle ECD$. Since N' is on the perpendicular bisectors of EC and CD, we have $\angle N'MC = \angle N'BC = 90^\circ$. Thus N' is on the circumcircle of $\triangle BMC$. Similarly, N' is on the circumcircle of $\triangle DMA$. As $N' \neq M$, we must have $N' = N$. So it suffices to compute R^2, where R is the circumradius of $\triangle ECD$.

   We can compute $K = [\triangle ECD]$ to be $21\sqrt{11}$ from Heron's formula, giving

   $$R = \frac{10 \cdot 14 \cdot 18}{4K} = \frac{30}{\sqrt{11}}.$$

   So $R^2 = \frac{900}{11}$, and the final answer is 90011.

5. Let AEF be a triangle with $EF = 20$ and $AE = AF = 21$. Let B and D be points chosen on segments AE and AF, respectively, such that BD is parallel to EF. Point C is chosen in the interior of triangle AEF such that $ABCD$ is cyclic. If $BC = 3$ and $CD = 4$, then the ratio of areas $\frac{[ABCD]}{[AEF]}$ can be written as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $100a + b$.

   Proposed by: Akash Das

   Answer: $\boxed{5300}$

   Solution 1: Rotate $\triangle ABC$ around A to $\triangle AB'C'$, such that B' is on segment AF. Note that as $BD \parallel EF$, $AB = AD$. From this, $AB' = AB = AD$, and $B' = D$. Note that

   $$\angle ADC' = \angle ABC = 180 - \angle ADC,$$

because $ABCD$ is cyclic. Therefore, C, D, and C' are collinear. Also, $AC' = AC$, and

$$\angle CAC' = \angle DAC + \angle C'AD = \angle DAC + \angle CAB = \angle EAF.$$

Thus, since $AE = AF$, $\triangle ACC' \sim \triangle AEF$. Now, we have

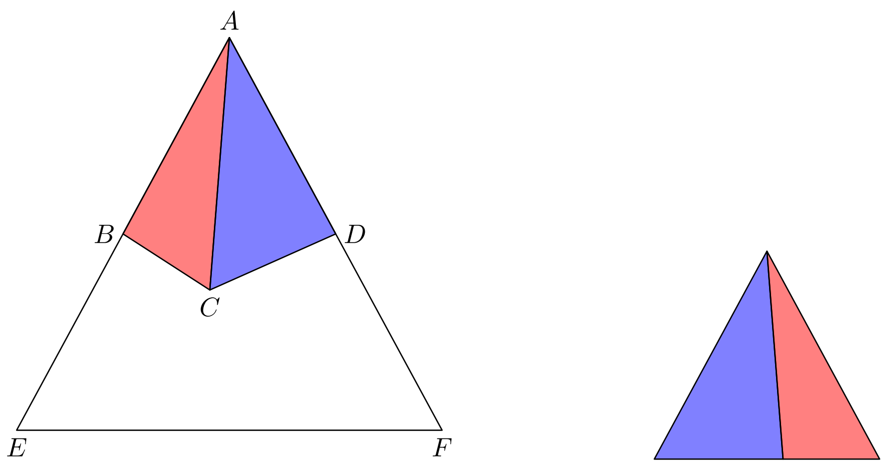
$$[ACC'] = [ACD] + [ADC'] = [ACD] + [ABC] = [ABCD].$$

But, $[ACC'] = \frac{CC'^2}{EF^2} \cdot [AEF]$, and we know that $CC' = CD + DC' = 4 + 3 = 7$. Thus,

$$\frac{[ABCD]}{[AEF]} = \frac{[ACC']}{[AEF]} = \frac{7^2}{20^2} = \frac{49}{400}.$$

The answer is $100 \cdot 49 + 400 = 5300$.

Solution 2:



Since BD is parallel to EF and $AE = AF$, we have $AB = AD$. Since $ABCD$ is cyclic, $\angle ABC + \angle ADC = 180^\circ$. Thus we can glue $\triangle ABC$ and $\triangle ADC$ as shown in the diagram above to create a triangle that is similar to $\triangle AEF$ and has the same area as $ABCD$. The base of this triangle has length $BC + CD = 3 + 4 = 7$, so the desired ratio is

$$\frac{7^2}{20^2} = \frac{49}{400}.$$

6. In triangle ABC, let M be the midpoint of BC, H be the orthocenter, and O be the circumcenter. Let N be the reflection of M over H. Suppose that $OA = ON = 11$ and $OH = 7$. Compute BC^2.

Proposed by: Milan Haiman

Answer: $\boxed{288}$

Solution: Let ω be the circumcircle of $\triangle ABC$. Note that because $ON = OA$, N is on ω. Let P be the reflection of H over M. Then, P is also on ω. If Q is the midpoint of NP, note that because

$$NH = HM = MP,$$

Q is also the midpoint of HM. Since $OQ \perp NP$, we know that $OQ \perp HM$. As Q is also the midpoint of HM,

$$OM = OH = 7.$$

With this,

$$BM = \sqrt{OB^2 - BM^2} = 6\sqrt{2},$$

and $BC = 2BM = 12\sqrt{2}$. Therefore, $BC^2 = 288$.

7. Let O and A be two points in the plane with $OA = 30$, and let Γ be a circle with center O and radius r. Suppose that there exist two points B and C on Γ with $\angle ABC = 90^\circ$ and $AB = BC$. Compute the minimum possible value of $\lfloor r \rfloor$.

   Proposed by: Hahn Lheem, Milan Haiman

   Answer: 12

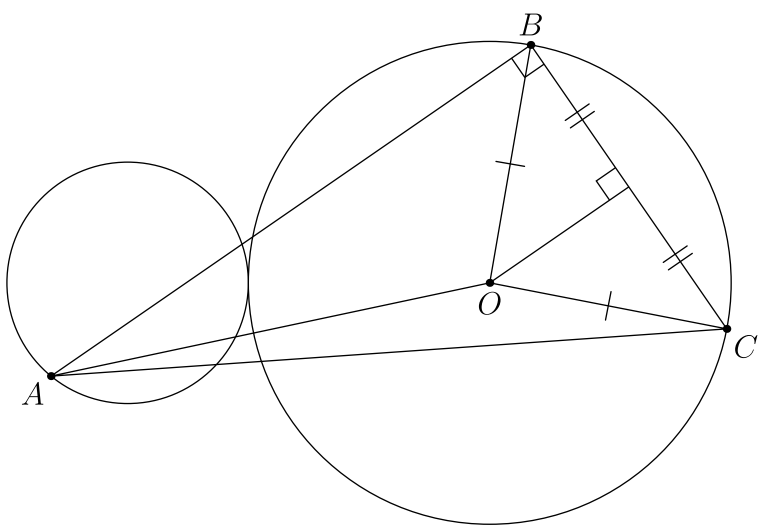
   Solution: Let f_1 denote a 45° counterclockwise rotation about point A followed by a dilation centered A with scale factor $1/\sqrt{2}$. Similarly, let f_2 denote a 45° clockwise rotation about point A followed by a dilation centered A with scale factor $1/\sqrt{2}$. For any point B in the plane, there exists a point C on Γ such that $\angle ABC = 90^\circ$ and $AB = BC$ if and only if B lies on $f_1(\Gamma)$ or $f_2(\Gamma)$. Thus, such points B and C on Γ exist if and only if Γ intersects $f_1(\Gamma)$ or $f_2(\Gamma)$. So, the minimum possible value of r occurs when Γ is tangent to $f_1(\Gamma)$ and $f_2(\Gamma)$. This happens when $r/\sqrt{2} + r = 30/\sqrt{2}$, i.e., when $r = \frac{30}{\sqrt{2}+1} = 30\sqrt{2} - 30$. Therefore, the minimum possible value of $\lfloor r \rfloor$ is $\lfloor 30\sqrt{2} - 30 \rfloor = 12$.

8. Two circles with radii 71 and 100 are externally tangent. Compute the largest possible area of a right triangle whose vertices are each on at least one of the circles.

   Proposed by: David Vulakh

   Answer: 24200

   Solution:



In general, let the radii of the circles be $r < R$, and let O be the center of the larger circle. If both endpoints of the hypotenuse are on the same circle, the largest area occurs when the hypotenuse is a diameter of the larger circle, with $[ABC] = R^2$.

If the endpoints of the hypotenuse are on different circles (as in the diagram above), then the distance from O to AB is half the distance from C to AB. Thus

$$[ABC] = 2[AOB] = AO \cdot OB \sin \angle AOB.$$

$AO \cdot OB$ and $\sin\angle AOB$ are simultaneously maximized when $AO \cdot OB = (2r+R) \cdot R$ and $m\angle AOB = 90°$, so the answer is $R^2 + 2Rr = 24200$.

9. Let $ABCD$ be a trapezoid with $AB \parallel CD$ and $AD = BD$. Let M be the midpoint of AB, and let $P \neq C$ be the second intersection of the circumcircle of $\triangle BCD$ and the diagonal AC. Suppose that $BC = 27, CD = 25$, and $AP = 10$. If $MP = \frac{a}{b}$ for relatively prime positive integers a and b, compute $100a + b$.

   Proposed by: Krit Boonsiriseth

   Answer: $\boxed{2705}$

   Solution 1: As $\angle PBD = \angle PCD = \angle PAB$, DB is tangent to (ABP). As $DA = DB$, DA is also tangent to (ABP). Let CB intersect (ABP) again at $X \neq B$; it follows that XD is the X-symmedian of $\triangle AXB$. As $\angle AXC = \angle DAB = \angle ADC$, X also lies on (ACD). Therefore $\angle PXB = \angle PAB = \angle PCD = \angle AXD$, so XP is the median of $\triangle AXB$, i.e. XP passes through M.

   Now we have $\triangle MPA \sim \triangle MBX$ and $\triangle ABX \sim \triangle BCD$. Therefore

$$\frac{MP}{AP} = \frac{MB}{XB} = \frac{AB}{2XB} = \frac{BC}{2DC} \implies MP = \frac{BC \cdot AP}{2DC} = \frac{27 \cdot 10}{2 \cdot 25} = \frac{27}{5}.$$

   Solution 2: Let E be the point such that $\square BDCE$ is a parallelogram. Then $\square ADCE$ is an isosceles trapezoid, therefore $\angle PAB = \angle CAB = \angle BED = \angle CDE$; this angle is also equal to $\angle PBD = \angle PCD$. Now, $\angle ABP = \angle ABD - \angle PBD = \angle BEC - \angle BED = \angle DEC$, therefore $\triangle PAB \sim \triangle CDE$. Let F be the midpoint of DE, which lies on BC because $\square BDCE$ is a parallelogram. It follows that $(\triangle PAB, M) \sim (\triangle CDE, F)$, therefore

$$\frac{MP}{AP} = \frac{FC}{DC} = \frac{BC}{2DC},$$

   and again this gives $MP = \frac{27}{5}$.

10. Acute triangle ABC has circumcircle Γ. Let M be the midpoint of BC. Points P and Q lie on Γ so that $\angle APM = 90°$ and $Q \neq A$ lies on line AM. Segments PQ and BC intersect at S. Suppose that $BS = 1$, $CS = 3$, $PQ = 8\sqrt{\frac{7}{37}}$, and the radius of Γ is r. If the sum of all possible values of r^2 can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b, compute $100a + b$.

   Proposed by: Jeffrey Lu

   Answer: $\boxed{3703}$

   Solution: Let A' be the A-antipode in Γ, let O be the center of Γ, and let $T := AA' \cap BC$. Note that A' lies on line PM. The key observation is that T is the reflection of S about M; this follows by the Butterfly Theorem on chords $\overline{PA'}$ and $\overline{AQ}$.

   Let $\theta := \angle AMP$ and $x = OT = OS$. Observe that $\cos\theta = \frac{PM}{AM} = \frac{PQ}{AA'} = \frac{PQ}{2r}$. We find the area of $\triangle AMA'$ in two ways. First, we have

$$\begin{aligned} 2[AMA'] &= AM \cdot MA \cdot \sin\theta \\ &= AM \cdot \tfrac{MB \cdot MC}{PM} \cdot \sin\theta \\ &= 4\tan\theta \\ &= 8r\sqrt{\tfrac{37}{448} - \tfrac{1}{4r^2}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} 2[AMA'] &= MT \cdot AA' \cdot \sin \angle OTM \\ &= 2r\sqrt{1 - \tfrac{1}{x^2}}. \end{aligned}$$

Setting the two expressions equal and squaring yields $\frac{37}{28} - \frac{4}{r^2} = 1 - \frac{1}{x^2}$. By Power of a Point, $3 = BS \cdot SC = r^2 - x^2$, so $x^2 = r^2 - 3$. Substituting and solving the resulting quadratic in r^2 gives $r^2 = \frac{16}{3}$ and $r^2 = 7$. Thus $\frac{a}{b} = \frac{37}{3}$, so $100a + b = 3703$.