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HMMT Spring 2021, March 06, 2021 — GUTS ROUND

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1. **[8]** Amelia wrote down a sequence of consecutive positive integers, erased one integer, and scrambled the rest, leaving the sequence below. What integer did she erase?

$$6, 12, 1, 3, 11, 10, 8, 15, 13, 9, 7, 4, 14, 5, 2$$

2. **[8]** Suppose there exists a convex n-gon such that each of its angle measures, in degrees, is an odd prime number. Compute the difference between the largest and smallest possible values of n.

3. **[8]** A semicircle with radius 2021 has diameter AB and center O. Points C and D lie on the semicircle such that $\angle AOC < \angle AOD = 90^\circ$. A circle of radius r is inscribed in the sector bounded by OA and OC and is tangent to the semicircle at E. If $CD = CE$, compute $\lfloor r \rfloor$.

4. **[8]** In a 3 by 3 grid of unit squares, an up-right path is a path from the bottom left corner to the top right corner that travels only up and right in steps of 1 unit. For such a path p, let A_p denote the number of unit squares under the path p. Compute the sum of A_p over all up-right paths p.

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5. **[9]** Let $m, n > 2$ be integers. One of the angles of a regular n-gon is dissected into m angles of equal size by $(m-1)$ rays. If each of these rays intersects the polygon again at one of its vertices, we say n is m-cut. Compute the smallest positive integer n that is both 3-cut and 4-cut.

6. **[9]** In a group of 50 children, each of the children in the group have all of their siblings in the group. Each child with no older siblings announces how many siblings they have; however, each child with an older sibling is too embarrassed, and says they have 0 siblings.

   If the average of the numbers everyone says is $\frac{12}{25}$, compute the number of different sets of siblings represented in the group.

7. **[9]** Milan has a bag of 2020 red balls and 2021 green balls. He repeatedly draws 2 balls out of the bag uniformly at random. If they are the same color, he changes them both to the opposite color and returns them to the bag. If they are different colors, he discards them. Eventually the bag has 1 ball left. Let p be the probability that it is green. Compute $\lfloor 2021p \rfloor$.

8. **[9]** Compute the product of all positive integers $b \geq 2$ for which the base b number 111111_b has exactly b distinct prime divisors.

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9. **[10]** Let AD, BE, and CF be segments sharing a common midpoint, with $AB < AE$ and $BC < BF$. Suppose that each pair of segments forms a 60° angle, and that $AD = 7$, $BE = 10$, and $CF = 18$. Let K denote the sum of the areas of the six triangles $\triangle ABC$, $\triangle BCD$, $\triangle CDE$, $\triangle DEF$, $\triangle EFA$, and $\triangle FAB$. Compute $K\sqrt{3}$.

10. **[10]** Let $a_1, a_2, \ldots, a_n$ be a sequence of distinct positive integers such that $a_1 + a_2 + \cdots + a_n = 2021$ and $a_1 a_2 \cdots a_n$ is maximized. If $M = a_1 a_2 \cdots a_n$, compute the largest positive integer k such that $2^k \mid M$.

11. **[10]** For each positive integer $1 \le m \le 10$, Krit chooses an integer $0 \le a_m < m$ uniformly at random. Let p be the probability that there exists an integer n for which $n \equiv a_m \pmod{m}$ for all m. If p can be written as $\frac{a}{b}$ for relatively prime positive integers a and b, compute $100a + b$.

12. **[10]** Compute the number of labelings $f : \{0,1\}^3 \to \{0,1,\ldots,7\}$ of the vertices of the unit cube such that

$$|f(v_i) - f(v_j)| \ge d(v_i, v_j)^2$$

for all vertices v_i, v_j of the unit cube, where $d(v_i, v_j)$ denotes the Euclidean distance between v_i and v_j.

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13. **[11]** A tournament among 2021 ranked teams is played over 2020 rounds. In each round, two teams are selected uniformly at random among all remaining teams to play against each other. The better ranked team always wins, and the worse ranked team is eliminated. Let p be the probability that the second best ranked team is eliminated in the last round. Compute $\lfloor 2021p \rfloor$.

14. **[11]** In triangle ABC, $\angle A = 2\angle C$. Suppose that $AC = 6$, $BC = 8$, and $AB = \sqrt{a} - b$, where a and b are positive integers. Compute $100a + b$.

15. **[11]** Two circles Γ_1 and Γ_2 of radius 1 and 2, respectively, are centered at the origin. A particle is placed at $(2, 0)$ and is shot towards Γ_1. When it reaches Γ_1, it bounces off the circumference and heads back towards Γ_2. The particle continues bouncing off the two circles in this fashion.

    If the particle is shot at an acute angle θ above the x-axis, it will bounce 11 times before returning to $(2, 0)$ for the first time. If $\cot\theta = a - \sqrt{b}$ for positive integers a and b, compute $100a + b$.

16. **[11]** Let $f : \mathbb{Z}^2 \to \mathbb{Z}$ be a function such that, for all positive integers a and b,

$$f(a,b) = \begin{cases} b & \text{if } a > b \\ f(2a,b) & \text{if } a \leq b \text{ and } f(2a,b) < a \\ f(2a,b) - a & \text{otherwise.} \end{cases}$$

    Compute $f(1000, 3^{2021})$.

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17. **[12]** Let k be the answer to this problem. The probability that an integer chosen uniformly at random from $\{1, 2, \ldots, k\}$ is a multiple of 11 can be written as $\frac{a}{b}$ for relatively prime positive integers a and b. Compute $100a + b$.

18. **[12]** Triangle ABC has side lengths $AB = 19$, $BC = 20$, and $CA = 21$. Points X and Y are selected on sides AB and AC, respectively, such that $AY = XY$ and XY is tangent to the incircle of $\triangle ABC$. If the length of segment AX can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers, compute $100a + b$.

19. **[12]** Almondine has a bag with N balls, each of which is red, white, or blue. If Almondine picks three balls from the bag without replacement, the probability that she picks one ball of each color is larger than 23 percent. Compute the largest possible value of $\left\lfloor \frac{N}{3} \right\rfloor$.

20. **[12]** Let $f(x) = x^3 - 3x$. Compute the number of positive divisors of

$$\left\lfloor f\left(f\left(f\left(f\left(f\left(f\left(f\left(f\left(\frac{5}{2}\right)\right)\right)\right)\right)\right)\right)\right)\right\rfloor,$$

where f is applied 8 times.

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21. **[14]** Bob knows that Alice has 2021 secret positive integers $x_1, \ldots, x_{2021}$ that are pairwise relatively prime. Bob would like to figure out Alice's integers. He is allowed to choose a set $S \subseteq \{1, 2, \ldots, 2021\}$ and ask her for the product of x_i over $i \in S$. Alice must answer each of Bob's queries truthfully, and Bob may use Alice's previous answers to decide his next query. Compute the minimum number of queries Bob needs to guarantee that he can figure out each of Alice's integers.

22. **[14]** Let E be a three-dimensional ellipsoid. For a plane p, let $E(p)$ be the projection of E onto the plane p. The minimum and maximum areas of $E(p)$ are 9π and 25π, and there exists a p where $E(p)$ is a circle of area 16π. If V is the volume of E, compute V/π.

23. **[14]** Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function such that $f(1) = 1$ and $f(2n)f(2n+1) = 9f(n)^2 + 3f(n)$ for all $n \in \mathbb{N}$. Compute $f(137)$.

24. **[14]** Let P be a point selected uniformly at random in the cube $[0, 1]^3$. The plane parallel to $x+y+z = 0$ passing through P intersects the cube in a two-dimensional region $\mathcal{R}$. Let t be the expected value of the perimeter of $\mathcal{R}$. If t^2 can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers, compute $100a + b$.

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25. **[16]** Let n be a positive integer. Claudio has n cards, each labeled with a different number from 1 to n. He takes a subset of these cards, and multiplies together the numbers on the cards. He remarks that, given any positive integer m, it is possible to select some subset of the cards so that the difference between their product and m is divisible by 100. Compute the smallest possible value of n.

26. **[16]** Let triangle ABC have incircle ω, which touches BC, CA, and AB at D, E, and F, respectively. Then, let ω_1 and ω_2 be circles tangent to AD and internally tangent to ω at E and F, respectively. Let P be the intersection of line EF and the line passing through the centers of ω_1 and ω_2. If ω_1 and ω_2 have radii 5 and 6, respectively, compute $PE \cdot PF$.

27. **[16]** Let P be the set of points
$$\{(x, y) \mid 0 \leq x, y \leq 25, x, y \in \mathbb{Z}\},$$
and let T be the set of triangles formed by picking three distinct points in P (rotations, reflections, and translations count as distinct triangles). Compute the number of triangles in T that have area larger than 300.

28. **[16]** Caroline starts with the number 1, and every second she flips a fair coin; if it lands heads, she adds 1 to her number, and if it lands tails she multiplies her number by 2. Compute the expected number of seconds it takes for her number to become a multiple of 2021.

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29. **[18]** Compute the number of complex numbers z with $|z| = 1$ that satisfy

$$1 + z^5 + z^{10} + z^{15} + z^{18} + z^{21} + z^{24} + z^{27} = 0.$$

30. **[18]** Let $f(n)$ be the largest prime factor of $n^2 + 1$. Compute the least positive integer n such that $f(f(n)) = n$.

31. **[18]** Roger initially has 20 socks in a drawer, each of which is either white or black. He chooses a sock uniformly at random from the drawer and throws it away. He repeats this action until there are equal numbers of white and black socks remaining.

    Suppose that the probability he stops before all socks are gone is p. If the sum of all distinct possible values of p over all initial combinations of socks is $\frac{a}{b}$ for relatively prime positive integers a and b, compute $100a + b$.

32. **[18]** Let acute triangle ABC have circumcenter O, and let M be the midpoint of BC. Let P be the unique point such that $\angle BAP = \angle CAM$, $\angle CAP = \angle BAM$, and $\angle APO = 90°$. If $AO = 53, OM = 28$, and $AM = 75$, compute the perimeter of $\triangle BPC$.

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33. **[20]** After the Guts round ends, HMMT organizers will collect all answers submitted to all 66 questions (including this one) during the individual rounds and the guts round. Estimate N, the smallest positive integer that no one will have submitted at any point during the tournament.

    An estimate of E will receive $\max(0, 24 - 4|E - N|)$ points.

34. **[20]** Let $f(n)$ be the largest prime factor of n. Estimate

$$N = \left\lfloor 10^4 \cdot \frac{\sum_{n=2}^{10^6} f\left(n^2 - 1\right)}{\sum_{n=2}^{10^6} f(n)} \right\rfloor .$$

    An estimate of E will receive $\max\left(0, \left\lfloor 20 - 20\left(\frac{|E-N|}{10^3}\right)^{1/3} \right\rfloor\right)$ points.

35. **[20]** Geoff walks on the number line for 40 minutes, starting at the point 0. On the nth minute, he flips a fair coin. If it comes up heads he walks $\frac{1}{n}$ in the positive direction and if it comes up tails he walks $\frac{1}{n}$ in the negative direction. Let p be the probability that he never leaves the interval $[-2, 2]$. Estimate $N = \lfloor 10^4 p \rfloor$.

    An estimate of E will receive $\max\left(0, \left\lfloor 20 - 20\left(\frac{|E-N|}{160}\right)^{1/3} \right\rfloor\right)$ points.

36. **[20]** A set of 6 distinct lattice points is chosen uniformly at random from the set $\{1, 2, 3, 4, 5, 6\}^2$. Let A be the expected area of the convex hull of these 6 points. Estimate $N = \lfloor 10^4 A \rfloor$.

    An estimate of E will receive $\max\left(0, \left\lfloor 20 - 20\left(\frac{|E-N|}{10^4}\right)^{1/3} \right\rfloor\right)$ points.