HMMT Spring 2021

March 06, 2021

Guts Round

1. **[8]** Amelia wrote down a sequence of consecutive positive integers, erased one integer, and scrambled the rest, leaving the sequence below. What integer did she erase?

$$6, 12, 1, 3, 11, 10, 8, 15, 13, 9, 7, 4, 14, 5, 2$$

Proposed by: Andrew Gu

Answer: $\boxed{16}$

Solution: The sequence of positive integers exactly contains every integer between 1 and 15, inclusive. 16 is the only positive integer that could be added to this sequence such that the resulting sequence could be reordered to make a sequence of consecutive positive integers. Therefore, Amelia must have erased the integer 16.

2. **[8]** Suppose there exists a convex n-gon such that each of its angle measures, in degrees, is an odd prime number. Compute the difference between the largest and smallest possible values of n.

Proposed by: Andrew Gu

Answer: $\boxed{356}$

Solution: We can't have $n = 3$ since the sum of the angles must be 180° but the sum of three odd numbers is odd. On the other hand, for $n = 4$ we can take a quadrilateral with angle measures $83^\circ, 83^\circ, 97^\circ, 97^\circ$.

The largest possible value of n is 360. For larger n we can't even have all angles have integer measure, and 179 happens to be prime.

So, the answer is $360 - 4 = 356$.

3. **[8]** A semicircle with radius 2021 has diameter AB and center O. Points C and D lie on the semicircle such that $\angle AOC < \angle AOD = 90^\circ$. A circle of radius r is inscribed in the sector bounded by OA and OC and is tangent to the semicircle at E. If $CD = CE$, compute $\lfloor r \rfloor$.

Proposed by: Hahn Lheem

Answer: $\boxed{673}$

Solution: We are given

$$m\angle EOC = m\angle COD$$

and

$$m\angle AOC + m\angle COD = 2m\angle EOC + m\angle COD = 90^\circ.$$

So $m\angle EOC = 30^\circ$ and $m\angle AOC = 60^\circ$. Letting the radius of the semicircle be R, we have

$$(R - r)\sin\angle AOC = r \Rightarrow r = \frac{1}{3}R,$$

so

$$\lfloor r \rfloor = \left\lfloor \frac{2021}{3} \right\rfloor = 673.$$

4. **[8]** In a 3 by 3 grid of unit squares, an up-right path is a path from the bottom left corner to the top right corner that travels only up and right in steps of 1 unit. For such a path p, let A_p denote the number of unit squares under the path p. Compute the sum of A_p over all up-right paths p.

   Proposed by: Freddie Zhao

   Answer: $\boxed{90}$

   Solution: Each path consists of 3 steps up and 3 steps to the right, so there are $\binom{6}{3} = 20$ total paths. Consider the sum of the areas of the regions above all of these paths. By symmetry, this is the same as the answer to the problem. For any path, the sum of the areas of the regions above and below it is $3^2 = 9$, so the sum of the areas of the regions above and below all paths is $9 \cdot 20 = 180$. Therefore, our final answer is $\frac{1}{2} \cdot 180 = 90$.

5. **[9]** Let $m, n > 2$ be integers. One of the angles of a regular n-gon is dissected into m angles of equal size by $(m-1)$ rays. If each of these rays intersects the polygon again at one of its vertices, we say n is m-cut. Compute the smallest positive integer n that is both 3-cut and 4-cut.

   Proposed by: Carl Schildkraut

   Answer: $\boxed{14}$

   Solution: For the sake of simplicity, inscribe the regular polygon in a circle. Note that each interior angle of the regular $n-$gon will subtend $n-2$ of the n arcs on the circle. Thus, if we dissect an interior angle into m equal angles, then each must be represented by a total of $\frac{n-2}{m}$ arcs. However, since each of the rays also passes through another vertex of the polygon, that means $\frac{n-2}{m}$ is an integer and thus our desired criteria is that m divides $n-2$.

   That means we want the smallest integer $n > 2$ such that $n-2$ is divisible by 3 and 4 which is just $12 + 2 = 14$.

6. **[9]** In a group of 50 children, each of the children in the group have all of their siblings in the group. Each child with no older siblings announces how many siblings they have; however, each child with an older sibling is too embarrassed, and says they have 0 siblings.

   If the average of the numbers everyone says is $\frac{12}{25}$, compute the number of different sets of siblings represented in the group.

   Proposed by: Vincent Bian

   Answer: $\boxed{26}$

   Solution: For $i \geq 1$, let a_i be the number of families that have i members in the group. Then, among each family with i children in the group, the oldest child will say $i-1$, and the rest will say 0. Thus, the sum of all the numbers said will be $a_2 + 2a_3 + 3a_4 + 4a_5 + \cdots = 50 \times \frac{12}{25} = 24$.

   Also because there are 50 children total, we know that $a_1 + 2a_2 + 3a_3 + \cdots = 50$. We can subtract these two equations to get $a_1 + a_2 + a_3 + \cdots = 50 - 24 = 26$.

7. **[9]** Milan has a bag of 2020 red balls and 2021 green balls. He repeatedly draws 2 balls out of the bag uniformly at random. If they are the same color, he changes them both to the opposite color and returns them to the bag. If they are different colors, he discards them. Eventually the bag has 1 ball left. Let p be the probability that it is green. Compute $\lfloor 2021p \rfloor$.

   Proposed by: Milan Haiman

   Answer: $\boxed{2021}$

   Solution: The difference between the number of green balls and red balls in the bag is always 1 modulo 4. Thus the last ball must be green and $p = 1$.

8. **[9]** Compute the product of all positive integers $b \geq 2$ for which the base b number 111111_b has exactly b distinct prime divisors.

Proposed by: Esha Bhatia

Answer: $\boxed{24}$

Solution: Notice that this value, in base b, is

$$\frac{b^6-1}{b-1} = (b+1)(b^2-b+1)(b^2+b+1).$$

This means that, if b satisfies the problem condition, $(b+1)(b^2-b+1)(b^2+b+1) > p_1 \dots p_b$, where p_i is the ith smallest prime.

We claim that, if $b \geq 7$, then $p_1 \dots p_b > (b+1)(b^2-b+1)(b^2+b+1)$. This is true for $b = 7$ by calculation, and can be proven for larger b by induction and the estimate $p_i \geq i$.

All we have to do is to check $b \in 2, 3, 4, 5, 6$. Notice that for $b = 6$, the primes cannot include $2, 3$ and hence we want $\frac{6^6-1}{5}$ to be divisible product of 6 primes the smallest of which is 5. However, $5 \cdot 7 \cdots \cdot 17 > \frac{6^6-1}{5}$, and by checking we rule out 5 too. All that is left is $\{2, 3, 4\}$, all of which work, giving us an answer of 24.

9. **[10]** Let AD, BE, and CF be segments sharing a common midpoint, with $AB < AE$ and $BC < BF$. Suppose that each pair of segments forms a 60° angle, and that $AD = 7$, $BE = 10$, and $CF = 18$. Let K denote the sum of the areas of the six triangles $\triangle ABC$, $\triangle BCD$, $\triangle CDE$, $\triangle DEF$, $\triangle EFA$, and $\triangle FAB$. Compute $K\sqrt{3}$.

Proposed by: Milan Haiman

Answer: $\boxed{141}$

Solution: Let M be the common midpoint, and let $x = 7, y = 10, z = 18$. One can verify that hexagon $ABCDEF$ is convex. We have

$$[ABC] = [ABM] + [BCM] - [ACM] = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{x}{2} \cdot \frac{y}{2} + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{y}{2} \cdot \frac{z}{2} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{x}{2} \cdot \frac{z}{2} = \frac{\sqrt{3}(xy + yz - zx)}{16}.$$

Summing similar expressions for all 6 triangles, we have

$$K = \frac{\sqrt{3}(2xy + 2yz + 2zx)}{16}.$$

Substituting x, y, z gives $K = 47\sqrt{3}$, for an answer of 141.

Remark: As long as hexagon $ABCDEF$ is convex, K is the area of this hexagon.

10. **[10]** Let $a_1, a_2, \dots, a_n$ be a sequence of distinct positive integers such that $a_1 + a_2 + \cdots + a_n = 2021$ and $a_1 a_2 \cdots a_n$ is maximized. If $M = a_1 a_2 \cdots a_n$, compute the largest positive integer k such that $2^k \mid M$.

Proposed by: Sheldon Kieren Tan

Answer: $\boxed{62}$

Solution: We claim that the optimal set is $\{2, 3, \cdots, 64\} \setminus \{58\}$. We first show that any optimal set is either of the form $\{b, b+1, b+2, \dots, d\}$ or $\{b, b+1, \dots, d\} \setminus \{c\}$, for some $b < c < d$.

Without loss of generality, assume that the sequence $a_1 < a_2 < \cdots < a_n$ has the maximum product. Suppose $a_{j+1} > a_j + 2$. Then, increasing a_j by 1 and decreasing a_{j+1} by 1 will increase the product

M, contradicting the assumption that the sequence has the optimal product. Thus, any "gaps" in the a_i can only have size 1.

Now, we show that there can only be one such gap. Suppose $a_{j+1} = a_j + 2$, and $a_{k+1} = a_k + 2$, for $j < k$. Then, we can increase a_j by 1 and decrease a_{i+1} by 1 to increase the total product. Thus, there is at most one gap, and the sequence a_i is of one of the forms described before.

We now show that either $b = 2$ or $b = 3$. Consider any set of the form $\{b, b+1, b+2, \dots, d\}$ or $\{b, b+1, \dots, d\} \setminus \{c\}$. If $b = 1$, then we can remove b and increase d by 1 to increase the product. If $b > 4$, then we can remove b and replace it with 2 and $b-2$ to increase the product. Thus, we have $b = 2, 3$, or 4.

Suppose $b = 4$. If the next element is 5, we can replace it with a 2 and a 3 to increase the product, and if the next element is 6, we can replace it with a 1, 2, and 3 without making the product any smaller. Thus, we can assume that either $b = 2$ or $b = 3$.

The nearest triangular number to 2021 is $2016 = 1+2+\cdots+64$. Using this, we can compute that if $b = 2$, our set must be $\{2, 3, \cdots, 64\} \setminus \{58\}$, leading to a product of $\frac{64!}{58}$. If $b = 3$, our set is $\{3, \cdots, 64\} \setminus \{56\}$, leading to a product of $\frac{64!}{2 \cdot 56}$.

Thus, the maximum product is $\frac{64!}{58}$. We now compute the highest power of 2 that divides this expression. 64! includes 32 elements that contribute at least one power of 2, 16 that contribute at least two powers of 2, and so on until the one element that contributes at least six powers of 2. This means the highest power of 2 that divides 64! is $32 + 16 + \cdots + 2 + 1 = 63$. Finally, dividing by 58 removes one of these powers of 2, making the answer 62.

11. **[10]** For each positive integer $1 \le m \le 10$, Krit chooses an integer $0 \le a_m < m$ uniformly at random. Let p be the probability that there exists an integer n for which $n \equiv a_m \pmod{m}$ for all m. If p can be written as $\frac{a}{b}$ for relatively prime positive integers a and b, compute $100a + b$.

Proposed by: Daniel Zhu

Answer: $\boxed{1540}$

Solution: Tuples of valid a_m correspond with residues mod $\operatorname{lcm}(1, 2, \dots, 10)$, so the answer is

$$\frac{\operatorname{lcm}(1, 2, \dots, 10)}{10!} = \frac{2^3 \cdot 3^2 \cdot 5 \cdot 7}{2^8 \cdot 3^4 \cdot 5^2 \cdot 7} = \frac{1}{1440}.$$

12. **[10]** Compute the number of labelings $f : \{0,1\}^3 \to \{0, 1, \dots, 7\}$ of the vertices of the unit cube such that

$$|f(v_i) - f(v_j)| \ge d(v_i, v_j)^2$$

for all vertices v_i, v_j of the unit cube, where $d(v_i, v_j)$ denotes the Euclidean distance between v_i and v_j.

Proposed by: Krit Boonsiriseth

Answer: $\boxed{144}$

Solution: Let $B = \{0,1\}^3$, let $E = \{(x, y, z) \in B : x + y + z \text{ is even}\}$, and let $O = \{(x, y, z) \in B : x + y + z \text{ is odd}\}$. As all pairs of vertices within E (and within O) are $\sqrt{2}$ apart, is easy to see that $\{f(E), f(O)\} = \{\{0, 2, 4, 6\}, \{1, 3, 5, 7\}\}$.

- There are two ways to choose $f(E)$ and $f(O)$; from now on WLOG assume $f(E) = \{0, 2, 4, 6\}$.
- There are 4! ways to assign the four labels to the four vertices in E.

- The vertex opposite the vertex labeled 0 is in O, and it must be labeled $3, 5$, or 7. It is easy to check that for each possible label of this vertex, there is exactly one way to label the three remaining vertices.

Therefore the total number of labelings is $2 \cdot 4! \cdot 3 = 144$.

13. **[11]** A tournament among 2021 ranked teams is played over 2020 rounds. In each round, two teams are selected uniformly at random among all remaining teams to play against each other. The better ranked team always wins, and the worse ranked team is eliminated. Let p be the probability that the second best ranked team is eliminated in the last round. Compute $\lfloor 2021p \rfloor$.

    Proposed by: Milan Haiman

    Answer: $\boxed{674}$

    Solution: In any given round, the second-best team is only eliminated if it plays against the best team. If there are k teams left and the second-best team has not been eliminated, the second-best team plays the best team with probability $\frac{1}{\binom{k}{2}}$, so the second-best team survives the round with probability

$$1 - \frac{1}{\binom{k}{2}} = 1 - \frac{2}{k(k-1)} = \frac{k^2-k-2}{k(k-1)} = \frac{(k+1)(k-2)}{k(k-1)}.$$

So, the probability that the second-best team survives every round before the last round is

$$\prod_{k=3}^{2021} \frac{(k+1)(k-2)}{k(k-1)},$$

which telescopes to

$$\frac{\frac{2022!}{3!} \cdot \frac{2019!}{0!}}{\frac{2021!}{2!} \cdot \frac{2020!}{1!}} = \frac{2022! \cdot 2019!}{2021! \cdot 2020!} \cdot \frac{2! \cdot 1!}{3! \cdot 0!} = \frac{2022}{2020} \cdot \frac{1}{3} = \frac{337}{1010} = p.$$

So,

$$\lfloor 2021p \rfloor = \left\lfloor \frac{2021 \cdot 337}{1010} \right\rfloor = \left\lfloor 337 \cdot 2 + 337 \cdot \frac{1}{1010} \right\rfloor = 337 \cdot 2 = 674.$$

14. **[11]** In triangle ABC, $\angle A = 2\angle C$. Suppose that $AC = 6$, $BC = 8$, and $AB = \sqrt{a} - b$, where a and b are positive integers. Compute $100a + b$.

    Proposed by: Freddie Zhao, Milan Haiman

    Answer: $\boxed{7303}$

    Solution: Let $x = AB$, and $\angle C = \theta$, then $\angle A = 2\theta$ and $\angle B = 180 - 3\theta$.

    Extend ray BA to D so that $AD = AC$. We know that $\angle CAD = 180 - 2\theta$, and since $\triangle ADC$ is isosceles, it follows that $\angle ADC = \angle ACD = \theta$, and so $\angle DCB = 2\theta = \angle BAC$, meaning that $\triangle BAC \sim \triangle BCD$.

    Therefore, we have

$$\frac{x+6}{8} = \frac{8}{x} \implies x(x+6) = 8^2$$

Since $x > 0$, we have $x = -3 + \sqrt{73}$. So $100a + b = 7303$.

15. **[11]** Two circles Γ_1 and Γ_2 of radius 1 and 2, respectively, are centered at the origin. A particle is placed at $(2,0)$ and is shot towards Γ_1. When it reaches Γ_1, it bounces off the circumference and heads back towards Γ_2. The particle continues bouncing off the two circles in this fashion.

    If the particle is shot at an acute angle θ above the x-axis, it will bounce 11 times before returning to $(2,0)$ for the first time. If $\cot\theta = a - \sqrt{b}$ for positive integers a and b, compute $100a+b$.

    Proposed by: Hahn Lheem

    Answer: $\boxed{403}$

    Solution: By symmetry, the particle must bounce off of Γ_2 at points that make angles of $60°$, $120°$, $180°$, $240°$, and $300°$ with the positive x-axis. Similarly, the particle must bounce off of Γ_1 at points that make angles of $30°$, $90°$, $150°$, $210°$, $270°$, and $330°$ with the positive x-axis.

    In particular, the first point that the ball touches on Γ_1 is $(\cos 30°, \sin 30°)$. So,

$$\cot\theta = \frac{2-\cos 30°}{\sin 30°} = 4-\sqrt{3}.$$

16. **[11]** Let $f : \mathbb{Z}^2 \to \mathbb{Z}$ be a function such that, for all positive integers a and b,

$$f(a,b) = \begin{cases} b & \text{if } a > b \\ f(2a,b) & \text{if } a \le b \text{ and } f(2a,b) < a \\ f(2a,b) - a & \text{otherwise.} \end{cases}$$

    Compute $f(1000, 3^{2021})$.

    Proposed by: Akash Das

    Answer: $\boxed{203}$

    Solution: Note that $f(a,b)$ is the remainder of b when divided by a. If $a > b$ then $f(a,b)$ is exactly b mod a. If instead $a \le b$, our "algorithm" doubles our a by n times until we have $a \times 2^n > b$. At this point, we subtract a^{2n-1} from $f(a \cdot 2^n, b)$ and iterate back down until we get $a > b - a \cdot k > 0$ and $f(a,b) = b - a \cdot k$ for some positive integer k. This expression is equivalent to $b - a \cdot k$ mod a, or b mod a.

    Thus, we want to compute 3^{2021} mod 1000. This is equal to 3 mod 8 and 78 mod 125. By CRT, this implies that the answer is 203.

17. **[12]** Let k be the answer to this problem. The probability that an integer chosen uniformly at random from $\{1, 2, \ldots, k\}$ is a multiple of 11 can be written as $\frac{a}{b}$ for relatively prime positive integers a and b. Compute $100a+b$.

    Proposed by: Carl Schildkraut, Hahn Lheem

    Answer: $\boxed{1000}$

    Solution: We write $k = 11q + r$ for integers q, r with $0 \le r < 11$. There are q multiples of 11 from 1 to k, inclusive, so our probability is $\frac{a}{b} = \frac{q}{11q+r}$. Let $d = \gcd(q,r) = \gcd(q, 11q+r)$, so that the fraction $\frac{q/d}{(11q+r)/d}$ is how we would write $\frac{q}{11q+r}$ in simplified form. Since we require that a and b be relatively prime, we find $a = \frac{q}{d}$ and $b = \frac{11q+r}{d}$.

    Plugging these into the equation $k = 100a+b$, we find $11q+r = 100\frac{q}{d} + \frac{11q+r}{d}$, or $d(11q+r) = 111q+r$. Since d divides r and $r \le 10$, we have $d \le 10$.

    If we test the case $d = 10$, our equation becomes $q = 9r$. Since $r = 10$ is the only valid value that is a multiple of d, we get $q = 90$ and $k = 1000$. 10 is, in fact, the gcd of q and r, so we have found that

$k = 1000$ satisfies the problem. Testing other values of d does not produce a valid answer.

18. **[12]** Triangle ABC has side lengths $AB = 19$, $BC = 20$, and $CA = 21$. Points X and Y are selected on sides AB and AC, respectively, such that $AY = XY$ and XY is tangent to the incircle of $\triangle ABC$. If the length of segment AX can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers, compute $100a + b$.

Proposed by: Hahn Lheem

Answer: $\boxed{6710}$

Solution: Note that the incircle of $\triangle ABC$ is the A-excenter of $\triangle AXY$. Let r be the radius of this circle. We can compute the area of $\triangle AXY$ in two ways:

$$\begin{aligned} K_{AXY} &= \frac{1}{2} \cdot AX \cdot AY \sin A \\ &= r \cdot (AX + AY - XY)/2 \\ \implies AY &= \frac{r}{\sin A} \end{aligned}$$

We also know that

$$\begin{aligned} K_{ABC} &= \frac{1}{2} \cdot 19 \cdot 21 \sin A \\ &= r \cdot (19 + 20 + 21)/2 \\ \implies \frac{r}{\sin A} &= \frac{19 \cdot 21}{60} = \frac{133}{20} \end{aligned}$$

so $AY = 133/20$.

Let the incircle of $\triangle ABC$ be tangent to AB and AC at D and E, respectively. We know that $AX + AY + XY = AD + AE = 19 + 21 - 20$, so $AX = 20 - \frac{133}{10} = \frac{67}{10}$.

19. **[12]** Almondine has a bag with N balls, each of which is red, white, or blue. If Almondine picks three balls from the bag without replacement, the probability that she picks one ball of each color is larger than 23 percent. Compute the largest possible value of $\lfloor \frac{N}{3} \rfloor$.

Proposed by: James Lin

Answer: $\boxed{29}$

Solution: If $k = \lfloor \frac{N}{3} \rfloor$, then the maximum possible probability is $\frac{6k^3}{(3k)(3k-1)(3k-2)}$. with equality when there are k balls of each of the three colors. Going from $3k \to 3k+1$ replaces $\frac{k}{3k-2} \to \frac{k+1}{3k+1}$, which is smaller, and going from $3k+1 \to 3k+2$ replaces $\frac{k}{3k-1} \to \frac{k+1}{3k+2}$, which is again smaller. For this to be larger than $\frac{23}{100}$, we find we need $0 > 7k^2 - 207k + 46$, and so $k = 29$ is the maximal value.

20. **[12]** Let $f(x) = x^3 - 3x$. Compute the number of positive divisors of

$$\left\lfloor f\left(f\left(f\left(f\left(f\left(f\left(f\left(f\left(\frac{5}{2}\right)\right)\right)\right)\right)\right)\right)\right)\right\rfloor,$$

where f is applied 8 times.

Proposed by: Akash Das

Answer: $\boxed{6562}$

Solution: Note that $f(y+\frac{1}{y}) = (y+\frac{1}{y})^3 - 3(y+\frac{1}{y}) = y^3 + \frac{1}{y^3}$. Thus, $f(2+\frac{1}{2}) = 2^3 + \frac{1}{2^3}$, and in general $f^k\left(2+\frac{1}{2}\right) = 2^{3^k} + \frac{1}{2^{3^k}}$, where f is applied k times. It follows that we just need to find the number of divisors of $\left\lfloor 2^{3^8} + \frac{1}{2^{3^8}} \right\rfloor = 2^{3^8}$, which is just $3^8 + 1 = 6562$.

21. **[14]** Bob knows that Alice has 2021 secret positive integers $x_1, \ldots, x_{2021}$ that are pairwise relatively prime. Bob would like to figure out Alice's integers. He is allowed to choose a set $S \subseteq \{1, 2, \ldots, 2021\}$ and ask her for the product of x_i over $i \in S$. Alice must answer each of Bob's queries truthfully, and Bob may use Alice's previous answers to decide his next query. Compute the minimum number of queries Bob needs to guarantee that he can figure out each of Alice's integers.

Proposed by: David Vulakh

Answer: $\boxed{11}$

Solution: In general, Bob can find the values of all n integers asking only $\lfloor \log_2 n \rfloor + 1$ queries.

For each of Alice's numbers x_i, let Q_i be the set of queries S such that $i \in S$. Notice that all Q_i must be nonempty and distinct. If there exists an empty Q_i, Bob has asked no queries that include x_i and has no information about its value. If there exist $i, j, i \neq j$ such that $Q_i = Q_j$, x_i and x_j could be interchanged without the answer to any query changing, so there does not exist a unique sequence of numbers described by the answers to Bob's queries (Alice can make her numbers distinct).

From the above, $\lfloor \log_2 n \rfloor + 1$ is a lower bound on the number of queries, because the number of distinct nonempty subsets of $\{1, \ldots, n\}$ is $2^n - 1$.

If Bob asks any set of queries such that all Q_i are nonempty and disjoint, he can uniquely determine Alice's numbers. In particular, since the values $x_1, \ldots, x_{2021}$ are relatively prime, each prime factor of x_i occurs in the answer to query S_j iff $j \in Q(i)$ (and that prime factor will occur in each answer exactly to the power with which it appears in the factorization of x_i). Since all $Q(i)$ are unique, all x_i can therefore be uniquely recovered by computing the product of the prime powers that occur exactly in the answers to queries $Q(i)$.

It is possible for Bob to ask $\lfloor \log_2 n \rfloor + 1$ queries so that each i is contained in a unique nonempty subset of them. One possible construction is to include the index i in the jth query iff the 2^{i-1}-value bit is set in the binary representation of j. So the answer is $\lfloor \log_2 2021 \rfloor + 1 = 11$.

22. **[14]** Let E be a three-dimensional ellipsoid. For a plane p, let $E(p)$ be the projection of E onto the plane p. The minimum and maximum areas of $E(p)$ are 9π and 25π, and there exists a p where $E(p)$ is a circle of area 16π. If V is the volume of E, compute V/π.

Proposed by: Daniel Zhu

Answer: $\boxed{75}$

Solution: Let the three radii of E be $a < b < c$. We know that $ab = 9$ and $bc = 25$.

Consider the plane p where projection $E(p)$ has area 9π. Fixing p, rotate E on the axis passing through the radius with length b until $E(p)$ has area 25π. The projection onto p will be an ellipse with radii b and r, where r increases monotonically from a to c.

By Intermediate Value Theorem, there must exist a circular projection with radius b. As the area of this projection is 16π, $b = 4$. Thus,

$$V = \frac{4}{3}\pi \cdot abc = \frac{4}{3} \cdot \frac{225}{4}\pi = 75\pi.$$

23. **[14]** Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function such that $f(1) = 1$ and $f(2n)f(2n+1) = 9f(n)^2 + 3f(n)$ for all $n \in \mathbb{N}$. Compute $f(137)$.

Proposed by: Sheldon Kieren Tan

Answer: $\boxed{2215.}$

Solution: Plugging in $n = 1$ gives $f(2)f(3) = 12$, therefore $(f(2), f(3)) = (2, 6)$ or $(3, 4)$. However, the former implies

$$f(4)f(5) \geq (6+1)(6+2) > 42 = 9 \cdot 2^2 + 3 \cdot 2,$$

which is impossible; therefore $f(2) = 3$ and $f(3) = 4$. We now show by induction with step size 2 that $f(2n) = 3f(n)$ and $f(2n+1) = 3f(n) + 1$ for all n; the base case $n = 1$ has already been proven.

Assume the statement is true for $n < 2k$. Applying the given and the inductive hypothesis, we have

$$f(4k)f(4k+1) = (3f(2k))(3f(2k)+1) = (9f(k))(9f(k)+1)$$
$$f(4k+2)f(4k+3) = (3f(2k+1))(3f(2k+1)+1) = (9f(k)+3)(9f(k)+4)$$

Let $x = f(4k+1)$. Since f is strictly increasing, this implies $x \geq \sqrt{f(4k)f(4k+1)} > 9f(k)$ and $x \leq \sqrt{f(4k+2)f(4k+3)} - 1 < 9f(k) + 3$. So $x = 9f(k) + 1$ or $x = 9f(k) + 2$. Since $9f(k) + 2$ does not divide $9f(k)(9f(k)+1)$, we must have $f(4k+1) = x = 9f(k) + 1$ and $f(4k) = 9f(k)$. A similar argument shows that $f(4k+2) = 9f(k)+3$ and $f(4k+3) = 9f(k)+4$, and this completes the inductive step.

Now it is a straightforward induction to show that f is the function that takes a number's binary digits and treats it as base 3. Since $137 = 10001001_2$ in binary, $f(137) = 10001001_3 = 3^7 + 3^3 + 1 = 2215$.

Remark: $137 = 2021_4$.

24. **[14]** Let P be a point selected uniformly at random in the cube $[0,1]^3$. The plane parallel to $x+y+z = 0$ passing through P intersects the cube in a two-dimensional region $\mathcal{R}$. Let t be the expected value of the perimeter of $\mathcal{R}$. If t^2 can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers, compute $100a + b$.

Proposed by: Michael Ren

Answer: $\boxed{12108}$

Solution: We can divide the cube into 3 regions based on the value of $x + y + z$ which defines the plane: $x + y + z < 1$, $1 \leq x + y + z \leq 2$, and $x + y + z > 2$. The two regions on the ends create tetrahedra, each of which has volume $1/6$. The middle region is a triangular antiprism with volume $2/3$.

If our point P lies in the middle region, we can see that we will always get the same value $3\sqrt{2}$ for the perimeter of $\mathcal{R}$.

Now let us compute the expected perimeter given that we pick a point P in the first region $x+y+z < 1$. If $x + y + z = a$, then the perimeter of $\mathcal{R}$ will just be $3\sqrt{2}a$, so it is sufficient to find the expected value of a. a is bounded between 0 and 1, and forms a continuous probability distribution with value proportional to a^2, so we can see with a bit of calculus that its expected value is $3/4$.

The region $x+y+z > 2$ is identical to the region $x+y+z < 1$, so we get the same expected perimeter. Thus we have a $2/3$ of a guaranteed $3\sqrt{2}$ perimeter, and a $1/3$ of having an expected $\frac{9}{4}\sqrt{2}$ perimeter, which gives an expected perimeter of $\frac{2}{3} \cdot 3\sqrt{2} + \frac{1}{3} \cdot \frac{9}{4}\sqrt{2} = \frac{11\sqrt{2}}{4}$. The square of this is $\frac{121}{8}$, giving an extraction of 12108.

25. **[16]** Let n be a positive integer. Claudio has n cards, each labeled with a different number from 1 to n. He takes a subset of these cards, and multiplies together the numbers on the cards. He remarks

that, given any positive integer m, it is possible to select some subset of the cards so that the difference between their product and m is divisible by 100. Compute the smallest possible value of n.

Proposed by: Carl Schildkraut

Answer: $\boxed{17}$

Solution: We require that $n \geq 15$ so that the product can be divisible by 25 without being even. In addition, for any $n > 15$, if we can acquire all residues relatively prime to 100, we may multiply them by some product of $\{1, 2, 4, 5, 15\}$ to achieve all residues modulo 100, so it suffices to acquire only those residues. For $n = 15$, we have the numbers $\{3, 7, 9, 11, 13\}$ to work with (as 1 is superfluous); these give only $2^5 = 32$ distinct products, so they cannot be sufficient. So, we must have $n \geq 17$, whence we have the numbers $\{3, 7, 9, 11, 13, 17\}$. These generators are in fact sufficient. The following calculations are motivated by knowledge of factorizations of some small numbers, as well as careful consideration of which sets of numbers we have and haven't used. It is also possible to simply write out a table of which residues relatively prime to 100 are included once each number is added, which likely involves fewer calculations.

First, consider the set $\{3, 11, 13, 17\}$. This set generates, among other numbers, those in $\{1, 11, 21, 31, 51, 61\}$. Since $\{7, 9\}$ generates $\{1, 7, 9, 63\}$, which spans every residue class mod 10 relatively prime to 10, we need only worry about

$$\{41, 71, 81, 91\} \times \{1, 7, 9, 63\}.$$

Since 41 can be generated as $3 \cdot 7 \cdot 13 \cdot 17$ and 91 can be generated as $7 \cdot 13$, we need not worry about these times 1 and 9, and we may verify

$$41 \cdot 7 \equiv 87 \equiv 11 \cdot 17, \; 91 \cdot 63 \equiv 33 \equiv 3 \cdot 11.$$

and

$$91 \cdot 7 \equiv 37 \equiv 3 \cdot 9 \cdot 11 \cdot 13 \cdot 17,$$

using the method we used to generate 49 earlier. So, we only need to worry about

$$\{71, 81\} \times \{1, 7, 9, 63\}.$$

We calculate

$$71 \equiv 7 \cdot 9 \cdot 17, \; 71 \cdot 9 \equiv 39 \equiv 3 \cdot 13, \; 71 \cdot 63 \equiv 73 \equiv 3 \cdot 7 \cdot 13,$$

each of which doesn't use 11, allowing us to get all of

$$\{71, 81\} \times \{1, 9, 63\},$$

so we are only missing $71 \cdot 7 \equiv 97$ and $81 \cdot 7 \equiv 67$. We find

$$97 \equiv 3 \cdot 9 \cdot 11$$

and

$$67 \equiv 3 \cdot 9 \cdot 13 \cdot 17,$$

so all numbers are achievable and we are done.

26. **[16]** Let triangle ABC have incircle ω, which touches BC, CA, and AB at D, E, and F, respectively. Then, let ω_1 and ω_2 be circles tangent to AD and internally tangent to ω at E and F, respectively. Let P be the intersection of line EF and the line passing through the centers of ω_1 and ω_2. If ω_1 and ω_2 have radii 5 and 6, respectively, compute $PE \cdot PF$.

Proposed by: Akash Das

Answer: $\boxed{3600}$

Solution: Let the centers of ω_1 and ω_2 be O_1 and O_2. Let DE intersect ω_1 again at Q, and let DF intersect ω_2 again at R. Note that since ω_1 and ω_2 must be tangent to AD at the same point (by equal tangents), so AD must be the radical axis of ω_1 and ω_2, so $RQEF$ is cyclic. Thus, we have

$$\angle O_1QR = \angle EQR - \angle O_1QE = 180^\circ - \angle EFD - \angle O_1EQ = 90^\circ$$

Thus, we have QR is tangent to ω_1, and similarly it must be tangent to ω_2 as well.

Now, note that by Monge's theorem on ω, ω_1, and ω_2, we have that P must be the intersection of the external tangents of ω_1 and ω_2. Since RQ is an external tangent, we have P, Q, and R are collinear. Thus, by power of a point, we have $PE \cdot PF = PR \cdot PQ$. Note that $PR = 10\sqrt{30}$ and $PQ = 12\sqrt{30}$. Thus, we have $PE \cdot PF = 3600$.

27. **[16]** Let P be the set of points

$$\{(x, y) \mid 0 \le x, y \le 25, x, y \in \mathbb{Z}\},$$

and let T be the set of triangles formed by picking three distinct points in P (rotations, reflections, and translations count as distinct triangles). Compute the number of triangles in T that have area larger than 300.

Proposed by: Andrew Lin, Haneul Shin

Answer: $\boxed{436}$

Solution: Lemma: The area of any triangle inscribed in an a by b rectangle is at most $\frac{ab}{2}$. (Any triangle's area can be increased by moving one of its sides to a side of the rectangle). Given this, because any triangle in T is inscribed in a 25×25 square, we know that the largest possible area of a triangle is $\frac{25^2}{2}$, and any triangle which does not use the full range of x or y-values will have area no more than $\frac{25 \cdot 24}{2} = 300$.

There are $4 \cdot 25 = 100$ triangles of maximal area: pick a side of the square and pick one of the 26 vertices on the other side of our region; each triangle with three vertices at the corners of the square is double-counted once. To get areas between $\frac{25 \cdot 24}{2}$ and $\frac{25 \cdot 25}{2}$, we need to pick a vertex of the square ($(0, 0)$ without loss of generality), as well as $(25, y)$ and $(x, 25)$. By Shoelace, this has area $\frac{25^2 - xy}{2}$, and since x and y must both be integers, there are $d(n)$ ways to get an area of $\frac{25^2 - n}{2}$ in this configuration, where $d(n)$ denotes the number of divisors of n.

Since we can pick any of the four vertices to be our corner, there are then $4d(n)$ triangles of area $\frac{25^2 - n}{2}$ for $1 \le n \le 25$. So, we compute the answer to be

$$\begin{aligned}
|P| &= 100 + 4(d(1) + \ldots + d(24)) \\
&= 4 \sum_{k \le 24} \left\lfloor \frac{24}{k} \right\rfloor \\
&= 100 + 4(24 + 12 + 8 + 6 + 4 + 4 + 3 + 3 + 2 \cdot 4 + 1 \cdot 12) \\
&= 436.
\end{aligned}$$

28. **[16]** Caroline starts with the number 1, and every second she flips a fair coin; if it lands heads, she adds 1 to her number, and if it lands tails she multiplies her number by 2. Compute the expected number of seconds it takes for her number to become a multiple of 2021.

Proposed by: Carl Schildkraut, Krit Boonsiriseth

Answer: $\boxed{4040}$

Solution: Consider this as a Markov chain on $\mathbb{Z}/2021\mathbb{Z}$. This Markov chain is aperiodic (since 0 can go to 0) and any number can be reached from any other number (by adding 1), so it has a unique stationary distribution π, which is uniform (since the uniform distribution is stationary).

It is a well-known theorem on Markov chains that the expected return time from a state i back to i is equal to the inverse of the probability π_i of i in the stationary distribution. (One way to see this is to take a length $n \to \infty$ random walk on this chain, and note that i occurs roughly π_i of the time.) Since the probability of 0 is $\frac{1}{2021}$, the expected return time from 0 to 0 is 2021.

After the first step (from 0), we are at 1 with probability $1/2$ and 0 with probability $1/2$, so the number of turns it takes to get from 1 to 0 on expectation is $2 \cdot 2021 - 2 = 4040$.

29. **[18]** Compute the number of complex numbers z with $|z| = 1$ that satisfy

$$1 + z^5 + z^{10} + z^{15} + z^{18} + z^{21} + z^{24} + z^{27} = 0.$$

Proposed by: Daniel Zhu

Answer: $\boxed{11}$

Solution: Let the polynomial be $f(z)$. One can observe that

$$f(z) = \frac{1 - z^{15}}{1 - z^5} + z^{15}\frac{1 - z^{15}}{1 - z^3} = \frac{1 - z^{20}}{1 - z^5} + z^{18}\frac{1 - z^{12}}{1 - z^3},$$

so all primitive 15th roots of unity are roots, along with -1 and $\pm i$.

To show that there are no more, we can try to find $\gcd(f(z), f(1/z))$. One can show that there exist a, b so that $z^a f(z) - z^b f(1/z)$ can be either of these four polynomials:

$$(1 + z^5 + z^{10})(1 - z^{32}), \quad (1 + z^5 + z^{10} + z^{15})(1 - z^{30}),$$
$$(1 + z^3 + z^6 + z^9 + z^{12})(z^{32} - 1), \quad (1 + z^3 + z^6 + z^9)(z^{30} - 1).$$

Thus any unit circle root of $f(z)$ must divide the four polynomials $(1 - z^{15})(1 - z^{32})/(1 - z^5)$, $(1 - z^{20})(1 - z^{30})/(1 - z^5)$, $(1 - z^{15})(1 - z^{32})/(1 - z^3)$, $(1 - z^{12})(1 - z^{30})/(1 - z^3)$. This implies that z must be a primitive kth root of unity, where $k \in \{1, 2, 4, 15\}$. The case $k = 1$ is clearly extraneous, so we are done.

30. **[18]** Let $f(n)$ be the largest prime factor of $n^2 + 1$. Compute the least positive integer n such that $f(f(n)) = n$.

Proposed by: Milan Haiman

Answer: $\boxed{89}$

Solution: Suppose $f(f(n)) = n$, and let $m = f(n)$. Note that we have $mn \mid m^2 + n^2 + 1$.

First we find all pairs of positive integers that satisfy this condition, using Vieta root jumping.

Suppose $m^2 + n^2 + 1 = kmn$, for some positive integer k. Considering this as a quadratic in m, let the other root (besides m) be m'. We have $m' + m = kn$, so m' is an integer. Also, $mm' = n^2 + 1$. So if $m > n$ then $m' \leq n$. So if we have a solution (m, n) we can find a smaller solution (n, m'). In particular, it suffices to find all small solutions to describe all solutions. A minimal solution must have $m = n$, which gives only $m = n = 1$. We have that $k = 3$.

Now the recurrence $a_0 = a_1 = 1$, $a_n + a_{n+2} = 3a_{n+1}$ describes all solutions with consecutive terms. In fact this recurrence gives precisely other Fibonacci number: $1, 1, 2, 5, 13, 34, 89, 233, \ldots$

Checking these terms gives an answer of 89.

31. **[18]** Roger initially has 20 socks in a drawer, each of which is either white or black. He chooses a sock uniformly at random from the drawer and throws it away. He repeats this action until there are equal numbers of white and black socks remaining.

Suppose that the probability he stops before all socks are gone is p. If the sum of all distinct possible values of p over all initial combinations of socks is $\frac{a}{b}$ for relatively prime positive integers a and b, compute $100a + b$.

Proposed by: Michael Diao

Answer: $\boxed{20738}$

Solution: Let b_i and w_i be the number of black and white socks left after i socks have been thrown out. In particular, $b_0 + w_0 = 20$.

The key observation is that the ratio $r_i = \frac{b_i}{b_i+w_i}$ is a martingale (the expected value of r_{i+1} given r_i is just r_i).

Suppose WLOG that $b_0 < w_0$ (we will deal with the case $b_0 = w_0$ later). Say that we stop at i if $b_i = 0$ or $b_i = w_i$. Then the expected value of r_i when we stop is

$$\frac{1}{2} \cdot p + 0 \cdot (1-p) = \frac{b_0}{b_0 + w_0}$$

This rearranges to $p = \frac{2b_0}{b_0+w_0}$.

Meanwhile, if $b_0 = w_0 = 10$, we can reduce to the case $b_1 = 9 < 10 = w_1$. Hence

$$\sum_{b_0=0}^{10} p = \left(\sum_{b_0}^{9} \frac{2b_0}{20}\right) + \frac{18}{19} = \frac{9}{2} + \frac{18}{19} = \frac{207}{38}.$$

32. **[18]** Let acute triangle ABC have circumcenter O, and let M be the midpoint of BC. Let P be the unique point such that $\angle BAP = \angle CAM$, $\angle CAP = \angle BAM$, and $\angle APO = 90^\circ$. If $AO = 53, OM = 28$, and $AM = 75$, compute the perimeter of $\triangle BPC$.

Proposed by: Jeffrey Lu

Answer: $\boxed{192}$

Solution: The point P has many well-known properties, including the property that $\angle BAP = \angle ACP$ and $\angle CAP = \angle BAP$. We prove this for completeness.

Invert at A with radius $\sqrt{AB \cdot AC}$ and reflect about the A-angle bisector. Let P' be the image of P. The angle conditions translate to

- P' lies on line AM
- P' lies on the line parallel to BC that passes through the reflection of A about BC (since P lies on the circle with diameter $\overline{AO}$)

In other words, P' is the reflection of A about M. Then $BP' \parallel AC$ and $CP' \parallel AB$, so the circumcircles of $\triangle ABP$ and $\triangle ACP$ are tangent to AC and AB, respectively. This gives the desired result. $\square$

Extend BP and CP to meet the circumcircle of $\triangle ABC$ again at B' and C', respectively. Then $\angle C'BA = \angle ACP = \angle BAP$, so $BC' \parallel AP$. Similarly, $CB' \parallel AP$, so $BCB'C'$ is an isosceles trapezoid. In particular, this means $B'P = CP$, so $BP + PC = BB'$. Now observe that $\angle ABP = \angle CAP = \angle BAM$, so if AM meets the circumcircle of $\triangle ABC$ again at A', then $AA' = BB'$. Thus the perimeter of $\triangle BPC$ is $BP + PC + BC = BB' + BC = AA' + BC$.

Now we compute. We have

$$BC = 2\sqrt{AO^2 - OM^2} = 2\sqrt{81 \cdot 25} = 90$$

and Power of a Point gives

$$MA' = \frac{BM^2}{AM} = \frac{45^2}{75} = 27.$$

Thus $AA' + BC = 75 + 27 + 90 = 192$.

33. **[20]** After the Guts round ends, HMMT organizers will collect all answers submitted to all 66 questions (including this one) during the individual rounds and the guts round. Estimate N, the smallest positive integer that no one will have submitted at any point during the tournament.

An estimate of E will receive $\max(0, 24 - 4|E - N|)$ points.

Proposed by: Carl Schildkraut

Answer: $\boxed{139}$

Solution: The correct answer was 139.

Remark: Until the end of the Guts round, no team had submitted 71 as the answer to any question. One team, however, submitted 71 as their answer to this question, increasing the answer up to 139.

34. **[20]** Let $f(n)$ be the largest prime factor of n. Estimate

$$N = \left\lfloor 10^4 \cdot \frac{\sum_{n=2}^{10^6} f\left(n^2 - 1\right)}{\sum_{n=2}^{10^6} f(n)} \right\rfloor.$$

An estimate of E will receive $\max\left(0, \left\lfloor 20 - 20\left(\frac{|E-N|}{10^3}\right)^{1/3} \right\rfloor\right)$ points.

Proposed by: Carl Schildkraut

Answer: $\boxed{18215}$

Solution: We remark that

$$f(n^2 - 1) = \max(f(n-1), f(n+1)).$$

Let X be a random variable that evaluates to $f(n)$ for a randomly chosen $2 \le n \le 10^6$; we essentially want to estimate

$$\frac{\mathbb{E}[\max(X_1, X_2)]}{\mathbb{E}[X_3]}$$

where X_i denotes a variable with distribution identical to X (this is assuming that the largest prime factors of $n-1$ and $n+1$ are roughly independent).

A crude estimate can be compiled by approximating that $f(n)$ is roughly 10^6 whenever n is prime and 0 otherwise. Since a number in this interval should be prime with "probability" $\frac{1}{\ln 10^6}$, we may replace each X_i with a Bernoulli random variable that is 1 with probability $\frac{1}{\ln 10^6} \sim \frac{1}{14}$ and 0 otherwise. This gives us an estimate of

$$\frac{1 \cdot \frac{2 \cdot 14 - 1}{14^2}}{\frac{1}{14}} = \frac{27}{14}.$$

However, this estimate has one notable flaw: $n-1$ and $n+1$ are more likely to share the same primality than arbitrarily chosen numbers, since they share the same parity. So, if we restrict our sums to only considering $f(n)$ for odd numbers, we essentially replace each X_i with a Bernoulli random variable with expectation $1/7$, giving us an estimate of $\frac{13}{7}$, good for 5 points.

This estimate can be substantially improved if we consider other possible factors, which increases the correlation between $f(n-1)$ and $f(n+1)$ and thus decreases one's estimate. The correct value of N is 18215.

35. **[20]** Geoff walks on the number line for 40 minutes, starting at the point 0. On the nth minute, he flips a fair coin. If it comes up heads he walks $\frac{1}{n}$ in the positive direction and if it comes up tails he walks $\frac{1}{n}$ in the negative direction. Let p be the probability that he never leaves the interval $[-2, 2]$. Estimate $N = \lfloor 10^4 p \rfloor$.

An estimate of E will receive $\max\left(0, \left\lfloor 20 - 20\left(\frac{|E-N|}{160}\right)^{1/3}\right\rfloor\right)$ points.

Proposed by: Michael Ren

Answer: $\boxed{8101}$

Solution: To estimate it by hand, we'll do casework on the most likely ways that Geoff will go past $+2$, and double the answer. If Geoff starts with one of the three sequences below, he will be past 2 or very close to 2:

$$(+,+,+,+),(+,+,+,-,+,+),(+,+,-,+,+,+).$$

The probability of one of these happening is $\frac{1}{16}+\frac{2}{64}=\frac{3}{32}$. This gives an estimate of $p=\frac{3}{16}$, which gives $E = 8125$ and earns 9 points.

We can justify throwing out other starting sequences as follows. For example, suppose we start with $(+,+,-,-)$. At this point we are at $\frac{11}{12}$. The variance of the rest of our random walk is

$$\sum_{n=5}^{40}\frac{1}{n^2}<\frac{\pi^2}{6}-1-\frac{1}{4}-\frac{1}{9}-\frac{1}{16}<0.25.$$

So, the standard deviation of the rest of our walk is bounded by 0.5, which is much less than the $\frac{13}{12}$ Geoff needs to go to get to $+2$. One can use similar estimates for other sequences to justify them as negligible. Furthermore, we can even use similar estimates to justify that if Geoff get close enough to $+2$, he is very likely to escape the interval $[-2, 2]$.

The exact value for p is 0.8101502670..., giving $N = 8101$.

36. **[20]** A set of 6 distinct lattice points is chosen uniformly at random from the set $\{1,2,3,4,5,6\}^2$. Let A be the expected area of the convex hull of these 6 points. Estimate $N = \lfloor 10^4 A \rfloor$.

An estimate of E will receive $\max\left(0, \left\lfloor 20 - 20\left(\frac{|E-N|}{10^4}\right)^{1/3}\right\rfloor\right)$ points.

Proposed by: Milan Haiman

Answer: $\boxed{104552}$

Solution: The main tools we will use are linearity of expectation and Pick's theorem. Note that the resulting polygon is a lattice polygon, and this the expected area A satisfies

$$A = I + \frac{B}{2} - 1,$$

where I is the expected number of interior points and B is the expected number of boundary points. We may now use linearity of expectation to write this as

$$A = -1 + \sum_{p\in\{1,2,\ldots,6\}^2} \mathbb{E}[X_p],$$

where X_p is 1 if the point is inside the polygon, 1/2 if the point is on the boundary, and 0 otherwise. Letting $f(p) = \mathbb{E}[X_p]$, we may write this by symmetry as

$$A = -1 + 4f(1,1) + 8f(1,2) + 8f(1,3) + 4f(2,2) + 8f(2,3) + 4f(3,3).$$

There are many ways to continue the estimation from here; we outline one approach. Since $X_{(1,1)}$ is 1/2 if and only if $(1, 1)$ is one of the selected points (and 0 otherwise), we see

$$f(1,1) = \frac{1}{12}.$$

On the other hand, we may estimate that a central point is exceedingly likely to be within the polygon, and guess $f(3, 3) \approx 1$. We may also estimate $f(1, y)$ for $y \in \{2, 3\}$; such a point is on the boundary if and only if $(1, y)$ is selected or $(1, z)$ is selected for some $z < y$ and for some $z > y$. The first event happens with probability 1/6, and the second event happens with some smaller probability that can be estimated by choosing the 6 points independently (without worrying about them being distinct); this works out to give the slight overestimate

$$f(1,2), f(1,3) \approx \frac{1}{8}.$$

From here, it is not so clear how to estimate $f(2, 2)$ and $f(2, 3)$, but one way is to make $f(x, y)$ somewhat linear in each component; this works out to give

$$f(2,2) \approx \frac{1}{4},\ f(2,3) \approx \frac{1}{2}.$$

(In actuality the estimates we'd get would be slightly higher, but each of our estimates for $f(x, y)$ up until this point have been slight overestimates.) Summing these up gives us an estimate of $A \approx \frac{31}{3}$ or $E = 103333$, which earns 10 points. The actual value of A is 10.4552776..., and so $N = 104552$.